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Non-linear Income Tax System in an Open Economy**

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## Abstract

This study analyzes the optimal shadow prices of inputs to the public sector in the presence of a non-linear income tax system in the standard two-factor Heckscher–Ohlin small open-economy model. In this model, there are two factors of production (unskilled labor and skilled labor) and two tradable goods (skilled labor- and unskilled labor-intensive goods). The two tradable goods are produced in the private sector by using two types of labor. A public good is produced by using the two types of tradable goods and the two types of labor. I show that the optimal shadow prices of the two types of labor in the public sector depend on whether there is a distortion in private production. On the other hand, the optimal shadow prices of tradable inputs do not depend on the presence of distortion in private production. In the second-best allocation, in which the incentive compatibility is binding, there should be distortion in the private sector. Thus, the optimal shadow prices of non-tradable factors of production depend on whether the economy is at the second-best equilibrium or not. On the other hand, the optimal shadow prices for tradable inputs are always equal to the international prices. Thus, for setting the optimal shadow prices in an open economy, it is important to distinguish whether the inputs are tradable goods or non-tradable factors and whether the economy is at the second-best equilibrium or not.

## 1 Introduction

In all developed countries, the share of the public sector is not trivial. Thus, how to set shadow prices for inputs of the public sector is important from the viewpoint of economic policy.

Shadow prices are the prices that the public sector uses to choose optimal combinations of inputs to produce a given level of public goods. In the theory of public economics, there is a famous “Production Efficiency Theorem” proved by Diamond and Mirrlees (1971). The theorem states that aggregate production must be efficient. In a closed economy, this implies that the marginal rate of transformation (MRT) of two inputs in the public sector should be equal to that in the private sector. Otherwise, it would be possible to increase the production of public goods or private goods without decreasing the production of other goods. Thus, the shadow prices of the public sector should be equal to the market prices of producers, since this is equal to the MRT in the private sector. In a small open economy, international prices are another MRT. Thus, in a small open economy in which the public sector uses two tradable goods and two non-tradable production factors, such as labor, the production efficiency theorem implies that the optimal shadow prices of those tradable goods are the international prices of those two tradable goods. For the same reason, the optimal shadow prices of two non-tradable factors of production are the market factor prices.

In the Diamond–Mirrlees framework, it is assumed that the government can impose a commodity tax on each good and each factor of production. This implicitly assumes that the government can impose different tax rates when there are skilled workers and unskilled workers and when they are imperfect substitutes. However, when the government cannot know the type of each worker, imposing different tax rates on different types of workers causes incentive problems. Thus, in such a situation, it is

difficult to impose different tax rates on different types of labor. This implies that the government needs to rely on a non-linear income tax system.

This study analyzes the optimal shadow prices of the public sector in the presence of an optimal non-linear income tax system, assuming there are two types of labor and that the government cannot know the type of each worker. Specifically, I consider the standard two-sector Heckscher–Ohlin small open-economy model with two factors and two tradable goods, in which the two types of production factors are skilled labor and unskilled labor and they are non-tradable and imperfect substitutes. The government imposes an optimal non-linear income tax system for redistribution. The two tradable goods are a skilled labor-intensive good and an unskilled labor-intensive good. The two sectors are a skilled labor-intensive sector and an unskilled labor-intensive sector. The public sector uses the two types of labor and the two tradable goods as inputs to produce a given level of a public good.

In this setting, I show that the rule of the optimal shadow prices of inputs of the public sector depends on the types of inputs used in the public sector and the presence of distortion in private production. Specifically, first I show that the optimal shadow prices of two types of labor in the public sector should not be equal to the market price of the two types of labor when the government imposes a tax or subsidy on private production. In addition, I show that a tax or subsidy on private production is optimal when the incentive compatibility is binding. In general, when the government implements a non-linear income tax system and redistributes income from skilled workers to unskilled workers, the incentive compatibility constraint is binding. This implies that the optimal shadow prices of two types of labor are not their market prices. On the other hand, when the government does not distort private production for some exogenous reason, then the government policy is not the second best, and the optimal shadow prices of the two types of labor should be equal to their market prices.

The optimal shadow prices of the two tradable goods always should be equal to the international prices, regardless of the presence of distortion in private production.

My results imply that the Diamond–Mirrlees production efficiency theorem holds for tradable goods but not for labor. This sharp difference can be understood as follows. Using labor as an input of production affects the wage rates of the two types of labor through the general equilibrium effect. Changing the wage rates would affect the incentive compatibility constraint. In the presence of a non-linear income system, the incentive compatibility constraint binds in most cases. Changing the incentive compatibility constraint affects the welfare of the economy at the first-order level. Thus, when the government uses labor as inputs of the public sector, it needs to consider the direct cost as well as the effect on the incentive compatibility constraint. This implies that although the international prices of tradable goods represent the true opportunity cost of using those goods, the market price of the two types of labor does not represent the true economic cost of those two types of labor in the presence of a non-linear income tax system. Thus, to choose the optimal level of inputs of public goods, the government needs to consider those differences.

The remainder of the paper is organized as follows. In Section 2, I discuss the related literature. In Section 2, I review the previous literature on the shadow prices of the public sector. In Section 3, I set up the model and analyze the optimal shadow prices. In Section 4, I provide a brief conclusion.

## **2 Related Literature**

Diamond and Mirrlees (1971), the seminal study in the literature, show that even in the presence of distorting tax, it is optimal to keep aggregate production efficient. This is a very important result in the public finance literature because it simplifies the rule of taxes and the shadow prices. It simply states that the optimal shadow prices are

the producer prices. Many studies investigate this issue and consider the conditions when this theorem does not apply or investigate when the underlying assumptions are not satisfied (Dasgupta and Stiglitz, 1972; Boadway, 1975; Christiansen, 1981; Drèze and Stern, 1990).

Regarding the optimal non-linear income tax, Mirrlees (1971) is the seminal study in the literature, while the model of Stiglitz (1982) makes the analysis of a non-linear income tax very operational. Several studies analyze the production efficiency in the presence of the non-linear income tax system, assuming that skilled labor and unskilled labor are imperfect substitutes (Naito, 1999; Blackorby and Brett, 2004; Gaube, 2005). Naito (1996), Guesnerie (2001), and Spector (2001), extend (Stiglitz, 1982) to a small open economy and analyze the issue of trade policy in the presence of a non-linear income tax, but they do not analyze the optimal shadow prices in an open economy.<sup>1</sup>

Closely related literature involves analysis of the marginal cost of public funds. Liu (2004, 2011) analyze the relationship between shadow prices and the marginal cost of public funds in detail.

### **3 Analysis**

#### **3.1 Setup of the Model**

##### Consumers

The basic model is an extension of the model developed by Stiglitz (1982) to a small open economy. This extension has been used by several authors (Naito (1996); Guesnerie (2001); Spector (2001)). In this economy, there are two types of labor, two tradable goods, and two private sectors. The two type of labor are skilled labor and unskilled labor. The two goods are a skilled labor-intensive good and an unskilled labor-

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<sup>1</sup>? analyze the issue of optimal regulation and trade in a similar model.

intensive good. The two sectors are a skilled labor-intensive sector and an unskilled labor-intensive sector. The public sector uses the two types of labor and the two tradable goods to produce public goods. I assume that the amount of public goods that needs to be produced is given. The international price of good 1 is one and the international price of good 2 is  $p^*$ . It is well known that a tariff is equivalent to a production subsidy and a commodity tax. Thus, we assume that, without loss of generality, the government imposes a commodity tax and production subsidy (tax) on good 2. Let  $q$  be the consumer price of good 2 and  $p$  be the producer price of good 2. Then, we obtain

$$q = p^* + t \quad \text{and} \quad p = p^* + \sigma \quad (1)$$

Skilled workers, denoted by subscript  $s$ , supply skilled labor and unskilled workers, denoted by subscript  $u$ , supply unskilled labor. For simplicity, we assume that the populations of skilled workers and unskilled workers equal  $N_s$  and  $N_u$ , respectively. We assume that skilled workers and unskilled worker has the same preference. In addition, we assume that the utility function of type  $i$  worker ( $i=s,u$ ) is denoted as  $u(c_{1i}, c_{2i}, l_i)$  and is a strictly quasi-concave function with respect to  $(c_{1i}, c_{2i}, l_i)$  where  $l_i$  is the labor supply of a worker of type  $i$ , and  $(c_{i1}, c_{i2})$  represents consumption of goods 1 and 2 by a worker of type  $i$ . We assume that good 1, good 2, and leisure are normal goods.<sup>2</sup> Next, I define the conditional indirect utility function of a type  $i$  worker as follows:

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<sup>2</sup>The normality assumption on goods and leisure is sufficient condition for the so-called “single-crossing property” and plays an important role in determining the equilibrium. For a “single-crossing property”, refer to footnote 6.

$$V^i(q, x_i; l_i) \equiv \max_{\{c_{1i}, c_{2i}\}} u(c_{1i}, c_{2i}; l_i) \quad (2)$$

$$\text{st. } c_{1i} + qc_{2i} = x_i, \quad l_i \text{ is given}$$

Let  $D_i(q, x, l)$  be the conditional demand for good 2 a type  $i$  worker. Then, from Roy's identity, we obtain

$$\frac{\partial V^i}{\partial q} = -\frac{\partial V^i}{\partial x} D_i(p, x; l) \quad (3)$$

### Producers

There are two industries  $F^1$  and  $F^2$  in the private sectors of this economy. The production function of each industry is concave and exhibits constant returns to scale. We assume that each industry uses skilled labor and unskilled labor, and that each sector produces output  $y_1$  and  $y_2$ . Thus,

$$y_1 = F^1(L_s^1, L_u^1), \quad y_2 = F^2(L_s^2, L_u^2), \quad (4)$$

where  $L_i^k$  is type  $i$  labor used in sector  $k$  ( $k = 1, 2$ ). Let  $w_s$  and  $w_u$  be wages for skilled workers and unskilled workers, respectively. Each industry maximizes its profit given the price of goods and wages. We assume that industry 2 is always unskilled labor intensive for any pair of  $\{w_s, w_u\}$ . Let  $c_k(w_s, w_u)$  be a cost function to produce one unit of good  $k$   $\{k = 1, 2\}$ . Since the economy is closed, we assume that both goods are produced in equilibrium without loss of generality. Then, perfect competition and constant returns to scale imply

$$c_1(w_s, w_u) = 1, \quad c_2(w_s, w_u) = p \quad (5)$$

By using Shephard's lemma, the factor demands are

$$L_s^k = y_k \frac{\partial c_k(w_s, w_u)}{\partial w_s}, \quad L_u^k = y_k \frac{\partial c_k(w_s, w_u)}{\partial w_u} \quad (k = 1, 2). \quad (6)$$

As for labor markets, labor is perfectly mobile among the two private sectors and one public sector. Let the amount of skilled and unskilled labor used in the public good production be  $L_s^g$  and  $L_u^g$ , respectively. The labor market equilibrium conditions are

$$N_s l_s = L_s^1 + L_s^2 + L_s^g, \quad N_u l_u = L_u^1 + L_u^2 + L_u^g \quad (7)$$

For a given price  $p$ , equation (5) determines  $w_s$  and  $w_u$  uniquely. Thus,  $w_s$ ,  $w_u$  and the ratio  $w_u/w_s \equiv \Omega$  can be written as a function of  $p$ :

$$w_s = w_s(p), \quad w_u = w_u(p), \quad \frac{w_u}{w_s} \equiv \Omega = \Omega(p). \quad (8)$$

Given the producer price  $p$ , wages are determined from equation (5). From the theorem of Stolper and Samuelson (1941), the effect of the change of price on wages is<sup>3</sup>

$$w_u'(p) > 0, \quad w_s'(p) < 0, \quad \Omega'(p) > 0. \quad (9)$$

### Non-linear Income Tax and Incentive Compatibility Constraints

The objective of the government is to achieve a Pareto-efficient allocation under the assumptions that the government cannot observe workers' types, but can observe each worker's total labor income. Therefore, in order to achieve a Pareto-efficient allocation, the government presents a menu of "tax contracts", so that individual workers self-select the contract that the government intended.<sup>4</sup> We define  $T(\cdot)$  as a tax

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<sup>3</sup>Intuitively, if the producer price of good 2 increases, then the output of good 2 will increase and the output of good 1 will decrease. Since sector 1 is unskilled labor intensive and sector 2 is skilled labor intensive, the wages of skilled labor must increase and the wages of unskilled labor must decrease to restore equilibrium in the labor market.

<sup>4</sup>It might be contended that we do not observe this type of tax system at all. However, "the revelation principle" proves that any tax system or any mechanism can be replicated by an incentive-compatible direct mechanism. Since real resource allocation is completely replicated by the incentive-compatible direct mechanism, we lose no generality by assuming this type of tax system. For a more detailed explanation about "the revelation principle", refer to Myerson (1979).

or subsidy function of the government and  $T_i$  as the amount of a tax or subsidy when the government observes the total income  $R_i$ . Then,  $T_i$  is

$$T_i \equiv T(R_i) . \quad (10)$$

From the definition of  $T_i$ , the net income  $x^i$  of a worker  $i$  when she earns  $R_i$  becomes

$$x_i = R_i - T_i . \quad (11)$$

When a type  $i$  worker earns  $R_j$ , her labor supply is  $\frac{R_j}{w_i}$ . The “tax contract” must satisfy incentive-compatibility constraints. These constraints for workers are<sup>5</sup>

$$V^s(q, x_s, \frac{R_s}{w_s}) \geq V^s(q, x_u, \frac{R_u}{w_s}) \quad (12)$$

$$V^u(q, x_u, \frac{R_u}{w_u}) \geq V^u(q, x_s, \frac{R_s}{w_u}) . \quad (13)$$

The first constraint means that a skilled worker has an incentive to work  $\frac{R_s}{w_s} = l_s$ , to report income  $R_s$ , and to receive a net income  $x_s$  instead of mimicking an unskilled worker, working  $\frac{R_u}{w_s} = \frac{w_u l_u}{w_s}$ , reporting income  $R_u = w_u l_u$ , and receiving a net income  $x_u$ . The second constraint is similar for unskilled workers.<sup>6</sup> If the government provides an incentive-compatible menu  $\{(R_u, x_u), (R_s, x_s)\}$ , then all workers choose the “tax contract ” that the government intended.

### Production Possibility Frontier

For the analysis in Section 3, it is useful to work on the production possibility frontier of the private sector conditional on a given total labor force in the private sector. Because technology is convex and factor intensity differs between the two

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<sup>5</sup>We allow workers to pay negative tax. This means that the government pays a subsidy to workers.

<sup>6</sup>Furthermore, if we draw indifference curves of  $V(q, x, \frac{R}{w_s})$  and  $V(q, x, \frac{R}{w_u})$  with respect to  $(x, R)$ , then the indifference curve of  $V(q, x, \frac{R}{w_s})$  crosses the indifference curve of  $V(q, x, \frac{R}{w_u})$  only once. In the literature of mechanism design, this is called a “single-crossing property ”.

sectors, the production possibility set is a strictly convex set. Thus, once the price ratio between good 1 and good 2 and the total labor force in the private sector are given, the amount of goods 1 and 2 produced is determined uniquely. Let  $L_s^{pri}$  and  $L_u^{pri}$  be the total skilled and unskilled labor used in the private sector. We write the output function of good 2 as

$$y_2 = Y_2(p; L_s^{pri}, L_u^{pri}) \quad (14)$$

$$\text{where } L_s^{pri} = L_s^1 + L_s^2 \text{ and } L_u^{pri} = L_u^1 + L_u^2$$

Since the production possibility set is strictly convex,

$$\frac{\partial Y_2(p; L_s^{pri}, L_u^{pri})}{\partial p} > 0 \quad (15)$$

$$\text{if } Y_2 > 0 .$$

From the Rybczynski theorem, if the skilled (unskilled) labor force in the private sector increases, then the production of the skilled labor-intensive (unskilled labor-intensive) good will increase, and the production of the unskilled labor-intensive (skilled labor-intensive) good will decrease, given producer price  $p$ . Since sector 2 is unskilled labor intensive, we obtain

$$\frac{\partial Y_2(p; L_s^{pri}, L_u^{pri})}{\partial L_s^{pri}} < 0, \quad \frac{\partial Y_2(p; L_s^{pri}, L_u^{pri})}{\partial L_u^{pri}} > 0 . \quad (16)$$

Note that from the Rybczynski theorem, we obtain

$$\frac{\partial Y_2(p; L_s^{pri}, L_u^{pri})}{\partial L_s^{pri}} < 0, \quad \frac{\partial Y_2(p; L_s^{pri}, L_u^{pri})}{\partial L_u^{pri}} > 0 . \quad (17)$$

Observe that  $\partial Y_2 / \partial L_s^g = (\partial Y_2 / \partial L_u^{pri})(\partial L_s^{pri} / \partial L_s^g)$  and  $\partial L_s^{pri} / \partial L_s^g = -1$ . Thus, we obtain  $\partial Y_2 / \partial L_s^g > 0$ . Similarly,  $\partial Y / \partial L_u^g < 0$ .

### Public Production

In order to examine the issue of production efficiency of public production, I assume that there is a public sector (public enterprise) in this economy that produces some given amount of public good using two types of labor and two tradable goods. Let  $g_1$  and  $g_2$  be the amount of tradable goods used in the public sector. Let  $G(L_s^g, L_u^g, g_1, g_2)$  be the production function of the public sector. I assume that the isoquant curve of the production function is strictly convex to the origin, so that the cost-minimizing choice of inputs is determined uniquely. The government commands the manager of the public sector to produce some given amount of public good  $\bar{G}$  by using the shadow prices  $z_1, z_2, z_s$  and  $z_u$ .<sup>7</sup> The necessary money to produce public good is transferred from the government.<sup>8</sup> The public sector minimizes its cost based on its shadow prices:

$$\min z_s L_s^g + z_u L_u^g + z_1 g_1 + z_2 g_2 \quad (18)$$

$$\text{s.t. } G(L_s^g, L_u^g, g_1, g_2) \geq \bar{G}$$

where  $\bar{G}$  is given

### Government Budget Constraint

The government budget constraint is:

$$N_s T_s + N_u T_u + t(N_s D(q, x_s) + N_u D(q, x_u)) - \sigma Y_2 \geq w_s L_s^g + w_u L_u^g + g_1 + p^* g_2 \quad (19)$$

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<sup>7</sup>In this study, without loss of generality, I assume that the level of public good is exogenous and fixed, since our interest is the choice of the shadow prices for a given level of public good that needs to be produced. It is always possible to consider the decision of the optimal choice of inputs in the public sector in two steps. First, the government chooses the optimal level of public good. Second, the government considers the optimal choice of inputs given that a certain level of public good needs to be produced. Thus, the current analysis focuses on the second step.

<sup>8</sup>I assume there is no asymmetric information between the public sector and the government to focus on the issue of production efficiency. It is standard to adopt this assumption when the issue is production efficiency of public production.

The first two terms are the revenue or subsidies by a progressive income tax system, and the third term is the tax revenue from a commodity tax.  $w_s L_{ps} + w_u L_{pu}$  is the cost of producing the public good. Although the government mandates the manager of the public sector to use  $(z_1, z_2, z_s, z_u)$  for choosing the optimal inputs of public production, the government needs to pay the market prices to purchase those goods and labor. Note that  $T_s = w_s l_s - x_s$ ;  $T_u = w_u l_u - x_u$ ;  $x_s = c_{1s} + (p^* + \tau)c_{2s}$ ;  $x_u = c_{1u} + (p^* + \tau)c_{2u}$ . Thus, we rewrite the budget constraint in the following way:

$$w_s(N_s l_s - L_s^g) + w_u(N_u l_u - L_u^g) + \tau(N_s c_2^s + N_u c_2^u) - \sigma Y_2 \geq N_s x_s + N_u x_u + g_1 + p^* g_2 \quad (20)$$

Equation (20) means that the value of production evaluated at the international price should be equal to the value of the consumption evaluated at the international price.

#### Optimal non-linear income tax system

The Pareto-efficient non-linear income tax system for a given level of a distortion parameter can be obtained by solving the following programming problem:

$$\max_{\{l^s, l^u, x_s, x_u, t, L_s^g, L_u^g, g_1, g_2\}} V^s(q, x_s, l^s)$$

subject to:

$$V^u(q, x_u, l^u) \geq \bar{U}^u, \quad (\text{MUC})$$

$$V^s(q, x_s, l^s) \geq V^s(q, x_u, \Omega(p)l^u), \quad (\text{ICS})$$

$$V^u(q, x_u, l^u) \geq V^u(q, x_s, \frac{l^s}{\Omega(p)}) \quad (\text{ICU})$$

$$w_s(N_s l_s - L_s^g) + w_u(N_u l_u - L_u^g) + t(N_s c_2^s + N_u c_2^u) - \sigma Y_2 \geq N_s x_s + N_u x_u + g_1 + p^* g_2 \quad (\text{BC})$$

$$G(L_s^g, L_u^g, g_1, g_2) \geq \bar{G} \quad (\text{PUB})$$

where  $q = p^* + \tau$  and  $p = p^* + \sigma$

where MUC is the utility constraint for unskilled workers. The Pareto-efficient allocation can be obtained by maximizing the utility of one type of worker subject to the constraint that the utility of the other type of worker is above some level  $\bar{U}^u$ .<sup>9</sup> ICS is the incentive compatibility constraint for skilled workers and ICU is the incentive compatibility constraint for unskilled workers. BC is the government budget constraint. The government chooses  $l_s$ ,  $l_u$ ,  $x_s$ , and  $x_u$ , so that the Pareto-efficient allocation is achieved subject to MUC, ICS, ICU, BC, and PUB. Let  $\mu_m, \mu_s, \mu_u, \lambda$  and  $\gamma$ . Let  $\mathcal{L}$  be the Lagrangian function. The first-order conditions of  $l_s$ ,  $l_u$ ,  $x_s$ , and  $x_u$  are written in the appendix. In the next subsection, I analyze the first-order conditions of  $L_s^g$  and  $L_u^g$ . To analyze this case, I consider two cases in which the first case is  $\sigma = 0$  and other is  $\sigma \neq 0$ .

### 3.2 Optimal relative shadow price of two types of labor in public production

The first-order conditions of  $L_s^g$  and  $L_u^g$  become as follows:

$$-\lambda w^s - \lambda \sigma \frac{\partial Y_2}{\partial L_s^g} + \gamma G_{L_s^g} = 0 \quad (21)$$

$$-\lambda w^s - \lambda \sigma \frac{\partial Y_2}{\partial L_s^g} + \gamma G_{L_s^g} = 0 \quad (22)$$

respectively. This implies that

$$\frac{G_{L_s^g}}{G_{L_u^g}} = \frac{w^s + \sigma \frac{\partial Y_2}{\partial L_s^g}}{w^u + \sigma \frac{\partial Y_2}{\partial L_u^g}} \quad (23)$$

Note that the left-hand side is the MRT of the public sector and is equal to the relative shadow price of the two types of labor,  $w_s/w_u$ . The right-hand side is equal to  $w_s/w_u$  when  $\sigma = 0$ . When  $\sigma > 0$ , the relative shadow wage of the public sector

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<sup>9</sup>Since controlling  $R$  is equivalent to controlling  $L$ , we use  $x$  and  $L$  as control variables.

$z_s/z_u$  is higher than  $w_s/w_u$ . This means that the public sector should hire less skilled workers and more unskilled workers than the efficient production case in which the relative shadow wage,  $z_s/z_u$ , equals  $w_s/w_u$ . When  $\sigma < 0$ , the public sector should hire more skilled workers and less unskilled workers than the efficient production case.

The above results show that when  $\sigma \neq 0$ , the production efficiency of the two types of labor fails, while when  $\sigma = 0$ , the production efficiency of the two types of labor holds.

### Optimal distortion in private production

In the previous two subsections, I show that if the government were to give a subsidy to sector 2, that is,  $\sigma > 0$ , the government should hire more unskilled workers and less skilled worker in the public sector than for the case of efficient production, and vice versa if  $\sigma < 0$ . Now, I show that the incentive-compatibility constraint of skilled workers is binding in the presence of a non-linear income tax system, a subsidy on the unskilled labor-intensive sector is optimal, that is,  $\sigma > 0$ . To observe why, consider a simple case where unskilled workers attempt to mimic skilled workers. This implies that the incentive-compatibility constraint of unskilled workers is binding. The first-order condition implies that

$$-\mu_s V_3^{su} \Omega' l^u + \lambda \{w'_s L_s^{pri} + w'_u L_u^{pri} - y_2 - \sigma \frac{\partial Y_2}{\partial p}\} = 0 \quad (24)$$

where  $V_3^{su}$  is defined as

$$V_3^{su} \equiv \frac{\partial V^s(q, x_u, w_u l_u / w_s)}{\partial (w_u l_u / w_s)}$$

Now, to calculate the values inside the bracket, we need to calculate  $w'_s L_s^{pri} + w'_u L_u^{pri}$ . In Appendix I, I show that  $w'_s L_s^{pri} + w'_u L_u^{pri} = y_2$ . Thus, the first-order condition becomes

$$-\mu_s V_3^{su} \Omega' l^u - \lambda \sigma \frac{\partial Y_2}{\partial p} = 0 \quad (25)$$

Note that  $V_3^{su} < 0$  due to the disutility of work.  $\mu_s > 0$  when the incentive-compatibility constraint of skilled workers is binding. In addition, from the Stolper–Samuelson effect,  $\Omega' > 0$ . From the shape of the production possibility frontier,  $\partial Y_2/\partial P > 0$ . This implies that  $\sigma > 0$ .

The intuition of  $\sigma > 0$  is clear. Starting from zero production distortion ( $\sigma = 0$ ), increasing  $\sigma$  will increase the wage of unskilled labor and decrease the wage of skilled labor. This has a first-order welfare effect when the incentive-compatibility constraint is binding. On the other hand, starting from zero distortion, a change of  $\sigma$  has a second-order effect on income. Thus, it is better to introduce a distortion.

The result  $\sigma > 0$  implies that the optimal shadow prices of two types of labor for the public sector are no longer the market wage rate in the private sector and the relative shadow wage of unskilled labor is lower than the relative market unskilled wage. Thus, the public sector should hire more unskilled labor than in the efficient production case.

#### Shadow prices of two tradable goods for the public sector

In the previous subsection, I have examined the optimal shadow price of two types of labor in the public sector. Now, I examine the shadow price of two tradable goods.

Note that in the optimization problem,  $g_1$  and  $g_2$  enter only the government budget constraint and this does not affect any relative price. Thus, the first-order conditions for  $g_1$  and  $g_2$  are

$$-\lambda + \gamma G_{g_1} = 0 \tag{26}$$

$$-\lambda p^* + \gamma G_{g_2} = 0 \tag{27}$$

Thus, for a pair of two output goods,  $\frac{G_{g_2}}{G_{g_1}} = p^*$ . Therefore, the relative shadow prices of two goods are the relative international prices. In contrast to the case of the two types of labor, this condition does not depend on whether there is production distortion in private production.

*Proposition*

*When the incentive compatibility of an optimal non-linear income tax system is binding, it is optimal to have a distortion in the private production and to subsidize the unskilled labor-intensive sector. In such a case, the relative shadow price of skilled labor to unskilled labor for the public sector should be higher than the pre-tax relative wage of skilled labor to unskilled labor in the private sector. As a result, production efficiency breaks down and the public sector should hire relatively more unskilled labor compared with the efficient production case. On the other hand, if the government is forced to use the non-second best policy and, as a result, there is no distortion in private production, the relative shadow price of two types of labor should be equal to the relative factor prices in the private sector and production efficiency holds. The shadow prices of the two output goods should always be equal to the international prices, regardless of the distortion in private production.*

At this point, readers might wonder why the production efficiency result of Diamond and Mirrlees (1971) does not hold for the factor of production. When the MRT of the public and private sectors initially is not equal, it is possible to increase the production of the public good or one of the private goods without decreasing the production of other goods by equating the MRT of the two sectors. A natural question is why such a change of production is not optimal. The answer to such a question lies in the factor price equalization theorem in international trade theory. The essence of the factor price equalization theorem is that there is a one-to-one relationship between factor prices and producer prices. To relax the incentive-compatibility constraint, it is necessary to introduce the production distortion in the private sector and to change the factor prices. However, when the MRT of the public and private sectors are equated, one factor moves from the public sector to the private sector and the other factor moves

from the private sector to the public sector. This implies that private production will change due to the change of factor allocation between public and private sectors. When production is already distorted owing to distortion, such a reallocation of the factors of production between the public and private sectors causes the first-order income loss. Therefore, it is not optimal to set the MRT of the public sector equal to that of the private sector.

There is an important message in the abovementioned proposition for setting the shadow price for public production. For tradable goods, the shadow prices should always equal international prices, regardless of the presence of other distortion. However, for such factors as labor, the shadow prices critically depend on the presence of other distortions, which is inevitable when the incentive compatibility constraint is binding.

#### **4 Conclusion and Summary**

In this study, I have analyzed the optimal shadow prices of the public sector in a small open economy in the presence of an optimal non-linear income tax system. I have shown that the optimal shadow prices of the public sector depend on the types of inputs used in the public sector and the presence of the distortion in the private sector, which is inevitable in the presence of an optimal non-linear income tax system. The results suggest that for setting the optimal shadow prices, we need to be careful about the effect on other distortions and the feedback mechanism from the public sector to the private sector. In this model, I assumed that non-tradable goods are labor. An interesting case would be the analysis of the optimal shadow prices in the presence of non-tradable goods (not factors), in which the public sector uses non-tradable goods as inputs. This is left for future research.

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## Appendices

### A.1 The first-order conditions of Lagrangian function

Let  $\mathcal{L}$  be the Lagrangian function. Then,  $\mathcal{L} = V^s(q, x_s, l^s) + \mu_m \{V^u(q, x_u, l^u) - \bar{U}^u\} + \mu_s \{V^s(q, x_s, l^s) - V^s(q, x_u, \Omega(p)l^u)\} + \mu_u \{V^u(q, x_u, l^u) - V^u(q, x_s, \frac{l^s}{\Omega(p)})\} + \lambda \{w_s(N_s l_s - L_s^g) + w_u(N_u l_u - L_u^g) + t(N_s c_2^s + N_u c_2^u) - \sigma Y_2 - N_s x_s - N_u x_u - g_1 - p^* g_2\} + \gamma \{G(L_s^g, L_u^g, g_1, g_2) - \bar{G}\}$ .

To calculate the first-order condition, define the partial derivative of the indirect utility function as follows:

$$\begin{aligned} V_1^{ii} &= \frac{\partial V^i(q, x_i, l_i)}{\partial q}, & V_1^{ij} &= \frac{\partial V^i(q, x_j, w_j l_j / w_i)}{\partial q}, \\ V_1^{ii} &= \frac{\partial V^i(q, x_i, l_i)}{\partial q}, & V_1^{ij} &= \frac{\partial V^i(q, x_j, w_j l_j / w_i)}{\partial x_j}, \\ V_3^{ii} &= \frac{\partial V^i(q, x_i, l_i)}{\partial l_i}, & V_3^{ij} &= \frac{\partial V^i(q, x_j, w_j l_j / w_i)}{\partial (w_j l_j / w_i)} \quad (j \neq i) \end{aligned}$$

In addition, define the partial derivative of the demand function as follows:

$$\begin{aligned} \frac{\partial D_i}{\partial x_i} &= \frac{\partial D(x_i, q, l_i)}{\partial x_i}, & \frac{\partial D_i}{\partial q} &= \frac{\partial D(x_i, q, l_i)}{\partial q} \\ \frac{\partial D_i}{\partial l_i} &= \frac{\partial D(x_i, q, l_i)}{\partial l_i} \end{aligned}$$

Then, the first-order conditions for  $l^s, l^u, x_s, x_u$ , and  $t$  become

$$\frac{\partial \mathcal{L}}{\partial x_s} = V_2^{ss} + \mu_s V_2^{ss} - \mu_u V_2^{us} - \lambda N_s + \lambda t N_s \frac{\partial D_s}{\partial x_s} = 0 \quad (28)$$

$$\frac{\partial \mathcal{L}}{\partial l_s} = V_3^{ss} + \mu_s V_3^{ss} - \mu_u V_3^{us} \frac{w_s}{w_u} + \lambda w_s N_s + \lambda t N_s \frac{\partial D_s}{\partial l_s} = 0 \quad (29)$$

$$\frac{\partial \mathcal{L}}{\partial x_u} = \mu_m V_2^{uu} + \mu_u V_2^{uu} - \mu_s V_2^{su} - \lambda N_u + \lambda t N_u \frac{\partial D_u}{\partial x_u} = 0 \quad (30)$$

$$\frac{\partial \mathcal{L}}{\partial l_u} = \mu_m V_3^{uu} + \mu_u V_3^{uu} - \mu_s V_3^{su} \frac{w_u}{w_s} + \lambda w_u N_u + \lambda t N_u \frac{\partial D_u}{\partial l_u} = 0 \quad (31)$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial t} &= V_1^{ss} + \mu_m V_1^{uu} + \mu_s V_1^{ss} - \mu_s V_1^{su} + \mu_u V_1^{uu} - \mu_u V_1^{us} \\ &+ \lambda \{ (N_s c_2^s + N_u c_2^u) + t \{ N_s \frac{\partial D_s}{\partial q} + N_u \frac{\partial D_u}{\partial q} \} \} = 0 \end{aligned} \quad (32)$$

## A.2 Proof of $w'_s L_s^{pri} + w'_u L_u^{pri} = y_2$

For given  $L_s^{pri}$  and  $L_u^{pri}$ , when  $\sigma$  changes, the change of  $w_s L_s^{pri} + w_u L_u^{pri}$  is equal to  $w'_s L_s^{pri} + w'_u L_u^{pri}$ . On the other hand, from an assumption of perfect competition and constant returns to scale, I obtain

$$w_s L_s^{pri} + w_u L_u^{pri} = y_1 + (p^* + \sigma) y_2 \quad (33)$$

On the other hand,  $w'_s L_s^{pri} + w'_u \bar{L}^u$  equals the change of  $y_1 + (p + \sigma)y_2$  in response to a change of  $\sigma$  when  $L_s^{pri}$  and  $L_u^{pri}$  are given. Now, for given  $L_s^{pri}$  and  $L_u^{pri}$ , consider the optimization problem

$$\begin{aligned} \pi(\sigma) &\equiv \max_{\{y_1, y_2\}} y_1 + (p^* + \sigma)y_2 & (34) \\ \text{subject to} & \quad (y_1, y_2) \in \bar{Y}(L_s^{pri}, \bar{L}^u), \end{aligned}$$

where  $\bar{Y}(L_s^{pri}, \bar{L}^u)$  is a production possibility set. Therefore, from the envelope theorem,

$$w'_s L_s^{pri} + w'_u L_u^{pri} = \frac{d\pi(\sigma)}{d\sigma} = y_2. \quad (35)$$