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Abstract

Regarding optimal design in the private value environment, there is an unsolved discrepancy in the literature regarding asymmetric auctions and auctions with endogenous participation; Literature on the former suggests that well-designed distortive mechanisms are optimal (revenue maximizing) assuming the bidding costs are negligible, while that on the latter insists that the mechanisms with free entry and no distortion are optimal provided that the potential bidders are *ex ante* symmetric.

This paper is the first attempt to reconcile the two views by establishing a model for asymmetric auctions with costly participation. The main findings are threefold; First, an optimal outcome is possible if and only if the mechanism is *ex post* efficient. Second, without any participation control, a coordination problem is likely in which only the *weak* bidders participate and the *strong* bidders stay out. Finally, there is an entry fee/subsidization scheme which, together with an *ex post* efficient mechanism, induces the optimal outcome as a unique equilibrium.

Key words: auctions, endogenous participation, asymmetric bidders
JEL classification: C72, D44, L22

1 Introduction

In high-valued asset or procurement auctions, the costs for preparing bids are typically non-trivial. The costs each bidder incurs prior to bidding range from information acquisition costs to transportation costs and even opportunity costs of awarding. Potential bidders who anticipate that bidding is unprofitable may hesitate to do so; therefore, designing a mechanism that accounts for the bidder's endogenous participation is crucial for the auction to be successful. The model of auction with endogenous entry has received

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comprehensive analysis motivated by such practical situations faced by the auctioneer (*e.g.*, McAfee and McMillan (1987), Engelbrecht-Wiggans (1993), Levin and Smith (1994), Kjerstad and Vagstad (2000), Ye (2004)). Furthermore, empirical studies of auctions with endogenous participation are growing following the development in the theoretical endogenous participation models.[1]

The notable insights presented in the existing literature, however, crucially depend on the assumption that all the potential bidders are *ex ante* identical. This strong assumption may result in analyses being restrictive. Consider the case, for example, where a limited number of bidders participate frequently but there are many other potential bidders who rarely enter the auction. The question of whether revenue-maximizing auctioneers should precommit to running a distortive auction to encourage entry by one-shot customers or give up promoting competition and simply set an entry fee to extract more surplus from the frequenters must be asked. The possibility that one-shot customers have a higher valuation for an item must also be examined. Furthermore, it must be determined whether more entries and stronger competition create higher revenue for the seller. The existing studies provide an ambiguous prediction for auctions with asymmetric potential bidders.[2]

In this research, we provide the first theoretical analysis for asymmetric auctions with endogenous participation. The model we establish is an extension of the model of auctions with costly participation, in which risk-neutral potential bidders randomize their participation in the auction. The bidders who actually enter incur a fixed participation cost and acquire their private information. In this formulation, we relax the symmetric assumption in the following manner. First, we suppose that potential bidders consist of two groups and that the bidders in each group participate in the auction with probability p_1 for one group and p_2 for the other.[3] The equilibrium we focus on in this paper is the *type-symmetric* mixed-strategy entry equilibrium which is constituted by a pair of probabilities (p_1, p_2). Second, we consider that the potential bidders in one group may be stronger than those in the other group, *i.e.*, the value distribution of a group of bidders stochastically dominates that of the remaining bidders.[4]

It is shown that there is at least one and typically multiple type-symmetric equilibria.

[1] See *e.g.* Athey and Haile (2006).

[2] The theoretical model provided by Pevnitskaya (2003a) characterizes an equilibrium with asymmetric bidders that assumes that different potential bidders have different risk attitudes. Pevnitskaya finds that some bidders who are less risk-averse tend to participate in auctions more frequently than others in equilibrium provided that each bidder's risk attitude is common knowledge. However, the equilibrium bidding function analyzed in her study is essentially symmetric.

[3] Although the model considers only two groups, it can be extended to a case involving three or more groups.

[4] There can be many forms of asymmetry. Our model does not restrict the form of asymmetry to stochastic dominance.

In some cases, a group of *weak* potential bidders enters with positive probability, and all the *strong* bidders stay out in equilibrium. We then demonstrate that, if the mechanism is *ex post* efficient, participation is always efficient in the sense that the expected marginal contribution of an additional participant to social surplus equals the marginal costs for participation. However, we also show that the "efficient entry" is not always optimal for the seller and society. Due to the stochastic entry process, some equilibria are more likely to end up with "too many" or "too few" bidders. These coordination costs are another source of efficiency loss, which makes all the efficient entry equilibria sub-optimal, except one. In the case of symmetric potential bidders, for example, a symmetric equilibrium creates the lowest revenue because of the highest coordination costs among equilibria.

It must be asked, therefore, how, when facing such a multiplicity problem, the auctioneer induces optimal participation. Our model suggests that, by introducing a participation fee/subsidization contingent on the realization of participation, the desired entry is always induced as a unique equilibrium regardless of the form of asymmetry. Moreover, the transfer scheme enables auctioneers to extract the entire surplus generated by the auction. Hence, sellers can implement the optimal entry by using a simple auction (English or second-price sealed-bid auction) with a well-chosen monetary transfer scheme.

These findings shed new light on the literature of auctions with asymmetric bidders as well as auctions with endogenous participation; First, our results contradict the theorems for an optimal design problem with a fixed number of bidders, which assert that a positive reservation price or some distortive allocation favoring a group of bidders improves revenue for the seller (*e.g.*, Riley and Samuelson (1981), Myerson (1981), Bulow and Roberts (1989), McAfee and McMillan (1989)). However, this argument ignores the point that the rent extraction from a group of potential bidders could depress their participation. Our results illustrate that *ex post* efficiency is essential for efficient entry and, thus, for optimal outcome, taking endogenous participation into account. It follows that any distortive mechanism entails the sub-optimal outcome in asymmetric auctions with potential bidders.

Second, the optimality of *ex post* efficient mechanisms challenges the ranking theorems. In asymmetric auctions, revenue ranking between first- and second-price mechanisms is generally ambiguous. This ambiguity, however, disappears, considering entry. We show that, by asymmetry, first-price auctions may attract more or fewer bidders, but the resulting excessive or deficient entry inflicts a greater burden on the seller, who bears all the participation costs. This story can easily be extended to the case of asymmetric auctions with affiliated private value (APV). Since affiliation gives greater advantage to second-price mechanisms, first-price mechanisms are still dominated by second-price mechanisms under the APV environment.

Third, our model extends the theoretical analysis of auctions with endogenous participation in several aspects. There are two groups of literature for auctions with costly participation, investigating either an asymmetric equilibrium (*e.g.*, McAfee and McMillan (1987)) or a symmetric equilibrium (*e.g.*, Levin and Smith (1994)). We provide a general theory that analyzes both simultaneously. This enables us to obtain a ranking method for social surplus and revenue across equilibria. Furthermore, the theorem is robust in cases of heterogeneous potential bidders. The symmetry assumption is somewhat disturbing since it is violated if it is common knowledge that some potential bidders are even slightly likely to have higher valuation for the item.

Finally, our results provide a theoretical background for the experimental analysis for auctions with endogenous participation. Pevnitskaya (2003b) observed evidence in laboratory experiments that some subjects are more likely to participate than the others. Since we show that the participation game is similar to a coordination game in which there are multiple equilibria, such an outcome is possible as an equilibrium.

The reminder of this paper is organized as follows; In Section 2 we describe the model. The Discussion is provided in Section 6. Section 7 is the conclusion. Proofs are presented in an appendix.

2 Model

Consider a risk-neutral seller auctions a single indivisible item to two groups of risk-neutral potential bidders with unit demand. Suppose that there are N_τ potential bidders in group $\tau \in \{1, 2\}$, and let $\mathcal{N}_\tau \equiv \{1, \cdots, N_\tau\}$ denote the index set of group τ potential bidders.

The transaction is described as a two-stage game. In the second stage, an auction takes place with $n = (n_1, n_2)$ actual bidders to allocate the item subject to an allocation rule set by the auctioneer. In the first stage, each potential bidder simultaneously makes his or her decision to incur a fixed participation cost c_τ and enter the auction game.

Throughout this paper, we suppose private values, *i.e.*, that one buyer's signal does not affect the other's preferences. Each bidder draws his or her own signal denoted by σ, which is, without loss of generality, uniformly distributed between 0 and 1. The value of the item for a bidder in group τ is captured by the valuation function $v_\tau : [0, 1] \to \Re_+$, which is strictly increasing and continuously differentiable. Finally, the seller's value for the item is normalized to equal zero.

The auction game consists of a set $B^i_\tau \in B$ of bids for each bidder, an allocation rule $x(\cdot|n) : B \to [0, 1]^n$, and a payment rule $k(\cdot|n) : B \to T^n$. If the ith bidder in group τ chooses a bid $b^i_\tau \in B^i_\tau$, then, given a bid profile $b = (b^1_1, \ldots b^{n_1}_1, b^1_2, \ldots b^{n_2}_2)$, the bidder obtains the item with probability equal $x^i_\tau(b|n) \in [0,1]$ and makes the expected payment $k^i_\tau(b|n)$ to the seller.

The entry game begins with the seller's announcement on the assignment rule $\xi = (\xi_1, \xi_2)$ with $\xi_\tau : \times_{n\in\mathcal{N}}\xi_\tau(b|n)$ and $\xi_\tau(b|n) = (\xi^1_\tau(b|n), \ldots, \xi^{n_\tau}_\tau(b|n))$, the payment rule $k = (k_1, k_2)$ with $k_\tau : \times_{n\in\mathcal{N}}k_\tau(b|n)$ and $k_\tau(b|n) = (k^1_\tau(b|n), \ldots, k^{n_\tau}_\tau(b|n))$, and the transfer schedule $y = (y_1, y_2)$ from the seller to the participants with $y_\tau : \times_{n\in\mathcal{N}}y_\tau(n)$ for $\tau = \{1, 2\}$. Given $\{\xi, k, y\}$, each of N_1+N_2 potential bidders simultaneously makes their entry decision by assigning a probability p^i_τ on his or her entry. Those who actually participate in the auction observe n, incur a participation cost c_τ, obtain a monetary transfer $y_\tau(n)$ and bid following a Nash bidding strategy $\beta^i_\tau(\sigma|n)$.

Now, let $\pi^i_\tau(n|\xi, k)$ denote the expected payoff of a bidder in group τ from the auction prior to drawing his or her private information. Then, the net gain of the bidder from participating in the auction, $u^i_\tau(n|\cdot)$, is given by

$$u^i_\tau(n|\xi, k, y) = \pi^i_\tau(n|\xi, k) + y_\tau(n) - c_\tau. \tag{1}$$

Throughout the paper, we focus on the class of transfer schemes Y such that $y_\tau(n)$ is decreasing in n. We also suppose that $\pi_\tau(n|x_\tau)$ is decreasing in n so that $u_\tau(n|x_\tau, y_\tau)$ is decreasing in n. Therefore, there are a set of pairs of numbers $n^* = (n^*_1, n^*_2)$ such that $u_\tau(n^*_\tau, n^*_{-\tau}|\cdot) \geq 0 > u_\tau(n^*_\tau + 1, n^*_{-\tau}|\cdot)$ for some $\tau = \{1, 2\}$.[5] To keep the model general, we do not assume that the participation costs c must be moderate. Therefore, for some $n^*_{-\tau}$, we could have $n^*_\tau = 0$ or $n^*_\tau > N_\tau$.

Hereafter, our analysis will proceed backward, beginning with the analysis of the second-stage auction game. After we obtain the equilibrium bidding strategy and the associated *ex ante* expected payoffs from the auction, we will investigate the entry decision in the first stage.

3 The Nash bidding strategy in asymmetric auctions

By the Revelation Principle, the asymmetric auction analyzed here can be described as the incentive-compatible (IC) direct-selling mechanism. Let $\sigma_\tau = (\sigma^1_\tau, \ldots, \sigma^{n_\tau}_\tau)$. Then, given a report profile on signals $\sigma = (\sigma_\tau, \sigma_{-\tau})$, a direct mechanism is characterized as an allocation rule $x(\sigma|n) = \{x^i_\tau(\sigma|n)\}_{i\in n_\tau, \tau\in\{1,2\}}$ and a payment rule $\lambda(\sigma) = \{\lambda^i_\tau(\sigma)\}_{i\in n_\tau, \tau\in\{1,2\}}$, where $x^i_\tau(t) \in [0, 1]$ is the probability with which the ith bidder in group τ obtains the item and $\lambda^i_\tau(t)$ is the expected payment the bidder makes to the auctioneer when σ is reported.

[5]Throughout, we assume that, for any function η, the first and second arguments for n_1 and n_2 are exchangeable, *i.e.*, $\eta(n_1, n_2|\cdot) \equiv \eta(n_2, n_1|\cdot)$

In our analysis, we focus on the class of the mechanism in which the assignment rule x and the payment λ for bidders in the same group are identical, respectively. In other words, we have $x^i_\tau \equiv x_\tau$, and $\lambda^i_\tau \equiv \lambda_\tau$ holds for all i. Hence, without using the superscript i, let w_τ be the conditional expected payoff a group τ bidder provided that his or her signal equals σ^i_τ. If the remaining $n-1$ participants report $(\sigma^{-i}_\tau, \sigma_{-\tau})$ truthfully, then the envelope integral formula makes w satisfy

$$w_\tau(\sigma^i_\tau|n, x_\tau) = w_\tau(0|n, x_\tau) + \int_0^{\sigma^i_\tau} \frac{d}{d\hat{\sigma}} v_\tau(\hat{\sigma}) x_\tau(\hat{\sigma}|n) d\hat{\sigma}.$$

Next, we derive the bidder's *ex ante* expected payoff from the asymmetric auction prior to drawing his or her signal σ^i_τ. Since σ is uniformly distributed between 0 and 1, the expected payoff $\pi_\tau(n|x_\tau) \equiv E[w_\tau(\sigma|\cdot)]$ is given by

$$\begin{aligned}\pi_\tau(n|x_\tau) &= \int_0^1 \int_0^{\sigma^i_\tau} \frac{d}{d\hat{\sigma}} v_\tau(\hat{\sigma}) x_\tau(\hat{\sigma}|n) d\hat{\sigma} d\sigma \\ &= \int_0^1 (1-\hat{\sigma}) \frac{d}{d\hat{\sigma}} v_\tau(\hat{\sigma}) x_\tau(\hat{\sigma}|n) d\hat{\sigma}, \qquad (2)\end{aligned}$$

where we normalize $w_\tau(0|n, x_\tau) = 0$. If there is a unique Nash bidding strategy denoted by $\beta(\sigma|n)$, then we have $x(\sigma|n) \equiv \xi(\beta(\sigma))$ for all $\sigma \in [0, 1]^n$. Thus $\pi_\tau(n|\xi_\tau) \equiv \pi_\tau(n|x_\tau)$ holds.

4 Type-symmetric entry equilibria

A type-symmetric equilibrium is an equilibrium in which all the bidders in the same group assign an identical probability on their participation. Suppose that bidders in group τ except i enter the auction with probability p_τ and each bidder in group $-\tau$ enters the auction with probability $p_{-\tau}$. In general, if $N_t - k$ potential bidders in group t enter the auction with probability p_t and $N_{-t} - \ell$ potential bidders in group $-t$ enter it with probability p_{-t}, then the probability that the number of actual entrants is equal to $\hat{n} = (\hat{n}_t, \hat{n}_{-t})$ is given by

$$P^{N_t-k, N_{-t}-\ell}_{\hat{n},p} \equiv \binom{N_t-k}{\hat{n}_t}\binom{N_{-t}-\ell}{\hat{n}_{-t}} [p_t]^{\hat{n}_t} [1-p_t]^{N_t-k-\hat{n}_t} [p_{-t}]^{\hat{n}_{-t}} [1-p_{-t}]^{N_{-t}-\ell-\hat{n}_{-t}}.$$

Therefore, provided that all the remaining potential bidders follow $p \in (p_1, p_2)$, the conditional expected gain of the ith bidder in group τ from participating in the auction $U^i_\tau(p_1, p_2|x_\tau, y_\tau)$ is written as

$$
\begin{cases}
U_1(p_1, p_2|x_1, y_1) \equiv \sum_{\hat{n}_1=0}^{N_1-1} \sum_{\hat{n}_2=0}^{N_2} P_{\hat{n},p}^{N_1-1,N_2} u_1(\hat{n}_1+1, \hat{n}_2|x_1, y_1) \\
U_2(p_1, p_2|x_2, y_2) \equiv \sum_{\hat{n}_1=0}^{N_1} \sum_{\hat{n}_2=0}^{N_2-1} P_{\hat{n},p}^{N_1,N_2-1} u_2(\hat{n}_1, \hat{n}_2+1|x_2, y_2),
\end{cases} \tag{3}
$$

where we omit N and $c = (c_1, c_2)$ on U_τ^i since these are exogenous throughout our research.

Given $\{x, y\}$, the ith bidder in group τ will randomize his participation if and only if $U_\tau(p_1, p_2|x_\tau, y_\tau) = 0$; otherwise, he will choose to enter or stay out as a pure strategy. To clarify, let $h_\tau : [0,1]^{N_\tau - 1} \times [0,1]^{N_{-\tau}} \to [0,1]$ be the best response entry decision of the ith potential bidder in group τ. Then, this best response function can be described as

$$
h_\tau(p_1, p_2|\cdot)
\begin{cases}
= 1 & \text{if } \{p_1, p_2|U_\tau(p_1, p_2|\cdot) > 0\} \\
= 0 & \text{if } \{p_1, p_2|U_\tau(p_1, p_2|\cdot) < 0\} \\
\in [0,1] & \text{if } \{p_1, p_2|U_\tau(p_1, p_2|\cdot) = 0\}.
\end{cases}
$$

A type-symmetric mixed strategy entry equilibrium is characterized by a pair of probabilities (p_1^*, p_2^*). Hence, the best response participation decision of the ith bidder in group τ, h_τ, must be equal to p_τ in equilibrium, implying that (p_1^*, p_2^*) satisfies

$$
h_\tau(p_1^*, p_2^*|\cdot) = p_\tau^*, \tag{4}
$$

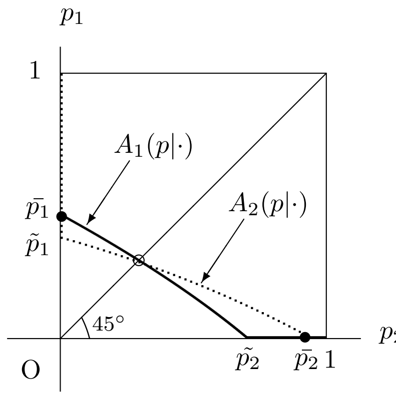
for all $\tau) \in \{1, 2\}$. In other words, if we define $A_\tau(p_1, p_2|\cdot) \equiv h_\tau(p_1, p_2|\cdot) - p_\tau$, then $p = (p_1, p_2)$ is a type-symmetric equilibrium if and only if $A_\tau(p|\cdot) = 0$ for all $\tau = \{1, 2\}$. Using this result, we have the following proposition which verifies the existence of such mixed-strategy equilibria in the asymmetric entry game.

Proposition 1. *There exists at least one mixed-strategy type-symmetric entry equilibrium in the participation game.*

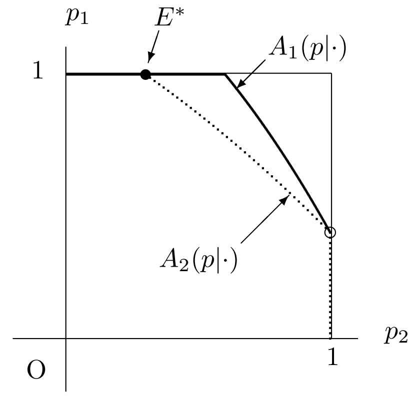
The proof given in the Appendix is absolutely in line with a regular proof of the existence of a mixed-strategy Nash equilibrium.

5 The property of the entry equilibria

The following figures depict A_τ as well as the mixed strategy entry equilibria as the intersections of A_τ on (p_1, p_2) space.



Figure(a)



Figure(b)

We define $\tilde{p}_\tau$ such that $U_2(\tilde{p}_1, 0|x_2, y_2) = 0$ and $U_1(0, \tilde{p_2}|x_1, y_1) = 0$. In addition, we define $\bar{p}_\tau$ such that $U_1(\bar{p}_1, 0|x_1, y_1) = 0$ and $U_2(0, \bar{p}_2|x_2, y_2) = 0$.

There are two types of mixed-strategy equilibria. To clarify, let $G_\tau(p|x_\tau, y_\tau) \equiv \partial p_1/\partial p_2|_{A_\tau=0}$ denote the absolute value of the slope of (10) on the p_1-p_2 square. Upon taking the total derivative of $A_\tau(\cdot)$ with respect to p_1 and p_2, one obtains $dA_\tau(p|x_\tau, y_\tau) = (\frac{\partial A_\tau}{\partial p_1})dp_1 + (\frac{\partial A_\tau}{\partial p_2})dp_2$.

Therefore, we have [6]

$$G_\tau(p|x_\tau, y_\tau) = \frac{\partial A_\tau/\partial p_2}{\partial A_\tau/\partial p_1} \tag{5}$$

if $p_\tau \in (0, 1)$.

Let an *"odd" mixed-strategy type-symmetric equilibrium* be the equilibrium such that $G_1(p^*|x, y) - G_2(p^*|x, y) \geq 0$, and let an *"even" mixed-strategy type-symmetric equilibrium* be the equilibrium such that $G_1(p^*|x, y) - G_2(p^*|x, y) < 0$.

Since $A_\tau(\cdot)$ is continuous and non-increasing, the number of *even* equilibria is always one fewer than the number of *odd* equilibria. Hence, by Proposition 1, there exists at

[6] We can compute G by

$$\partial A_1/\partial p_2 = N_2 \cdot \sum_{\hat{n}_1=0}^{N_1-1} \sum_{\hat{n}_2=0}^{N_2-1} P_{\hat{n},p}^{N_1-1,N_2-1} \left[u_1(\hat{n}_1+1, \hat{n}_2+1|\cdot) - u_1(\hat{n}_1+1, \hat{n}_2|\cdot)\right],$$

$$\partial A_1/\partial p_1 = (N_1-1) \cdot \sum_{\hat{n}_1=0}^{N_1-2} \sum_{\hat{n}_2=0}^{N_2} P_{\hat{n},p}^{N_1-2,N_2} \left[u_1(\hat{n}_1+2, \hat{n}_2|\cdot) - u_1(\hat{n}_1+1, \hat{n}_2|\cdot)\right],$$

$$\partial A_2/\partial p_2 = (N_2-1) \cdot \sum_{\hat{n}_1=0}^{N_1} \sum_{\hat{n}_2=0}^{N_2-2} P_{\hat{n},p}^{N_1,N_2-2} \left[u_2(\hat{n}_1, \hat{n}_2+2|\cdot) - u_2(\hat{n}_1, \hat{n}_2+1|\cdot)\right],$$

$$\partial A_2/\partial p_1 = N_1 \cdot \sum_{\hat{n}_1=0}^{N_1-1} \sum_{\hat{n}_2=0}^{N_2-1} P_{\hat{n},p}^{N_1-1,N_2-1} \left[u_2(\hat{n}_1+1, \hat{n}_2+1|\cdot) - u_2(\hat{n}_1+1, \hat{n}_2|\cdot)\right].$$

least one *odd* equilibrium and an *even* equilibrium exists if and only if there are multiple equilibria. Furthermore, if all potential bidders are identical and $y = \mathbf{0}$, the symmetric equilibrium is always *even* since $G_1(\rho, \rho) = N_2/(N_1 - 1)$ and $G_2(\rho, \rho) = (N_2 - 1)/N_1$ for any $\rho \in (0, 1)$.

To seek strategic interaction in the participation equilibrium, it is convenient to formulate the relative strength between the two groups by the difference in their expected gain from participation as follows:

Definition 1. *The group* 1 *potential bidders are* ***more profitable*** *than the group 2 potential bidders if and only if i)* $u_1(n_1 + 1, n_2|x_1, y_1) \geq u_2(n_1, n_2 + 1|x_2, y_2)$ *holds for some* x *and* y *and ii)* $u_1(n_1, n_1|x_1, y_1) \geq u_2(n_1, n_2|x_2, y_2)$ *holds for such* x *and* y*.*

Condition i) implies that the *ex ante* payoff of a particular bidder is monotonically increased by the change of his or her type from 2 to 1, whereas ii) implies that the *ex ante* payoff of a type 1 potential bidder is always greater than that of a type 2 bidder in the auction. If the mechanisms are *ex post* efficient, a more profitable bidder is equivalent to a strong bidder in the sense that his or her value distribution stochastically dominates the value distribution of a weak bidder.[7] Then, one obtains the following lemma about an equilibrium participation decision.

Proposition 2. *Suppose that a potential bidder in one group is* ***more profitable*** *than a potential bidder in the other group for some mechanism and transfer scheme. Then, the probability with which a less profitable bidder enters the auction is greater than the probability with which a more profitable bidder enters the auction if each potential bidder is indifferent between participating and staying out in equilibrium.*

See the Appendix for proof.

A symmetric equilibrium with *ex ante* identical bidders corresponds to the special case in which the equations in condition i) and ii) hold with equality, as shown in Figure (a). On the other hand, if asymmetry between two groups is sufficiently large, no *even* equilibrium

[7] If a mechanism is *ex post* efficient, we have

$$\begin{aligned} \pi_1(n_1+1, n_2|x) &= \int_v (1-F_1(v))(F_1(v))^{n_1}(F_2(v))^{n_2} dv, \\ \pi_2(n_1, n_2+1|x) &= \int_v (1-F_2(v))(F_1(v))^{n_1}(F_2(v))^{n_2} dv, \end{aligned}$$

for any n. Hence, $F_1(v) \leq F_2(v)$ for any v implies $\pi_1(n_1+1, n_2|x) \geq \pi_2(n_1, n_2+1|x)$. In addition, under the *ex post* efficient mechanism, we have

$$\begin{aligned} \pi_1(n_1, n_2|x) &= \int_v (1-F_1(v))(F_1(v))^{n_1-1}(F_2(v))^{n_2} dv \\ &= \int_v [F_2(v) - F_1(v)F_2(v)](F_1(v))^{n_1-1}(F_2(v))^{n_2-1} dv \\ \pi_2(n_1, n_2|x) &= \int_v (1-F_2(v))(F_1(v))^{n_1}(F_2(v))^{n_2-1} dv \\ &= \int_v [F_1(v) - F_1(v)F_2(v)](F_1(v))^{n_1-1}(F_2(v))^{n_2-1} dv \end{aligned}$$

for any n. Thus, if $F_1(v) \leq F_2(v)$ for any v, then $\pi_1(n_1, n_2|x) \geq \pi_2(n_1, n_2|x)$.

is likely. The marginal case is shown in Figure (b), in which an *even* equilibrium will disappear if group 1 bidders become more profitable.[8]

If potential bidders across groups are identical and c is moderate, there exist two additional asymmetric equilibria. The asymmetric equilibrium analyzed by McAfee and McMillan (1987), Engelbrecht-Wiggans (1993) is the case in which $p^*_\tau = 1$ and $p^*_{-\tau} = 0$ for some τ. Depending on N_1 and N_2, there could be many other asymmetric equilibria *e.g.*, $p^*_1 = 1$ and $p^*_2 \in (0, 1)$, as shown in Figure (b), where group 1 bidders obtain positive expected rents.

The auctioneer has no reason to keep the bidders obtaining strictly positive rents. The following lemma shows that full extraction of rents is trivially possible by an entry fee.

Lemma 1. *For any p, there exists a constant participation fee schedule $y^0_{p,x,y} = (y^0_{1,p,x_1,y_1}, y^0_{2,p,x_2,y_2}) \in \Re^2$ such that full extraction of rents is possible for some $x \in X$ and $y \in Y$. Moreover, $y^0_{p,x,y}$ implements the rent extraction by holding p constant.*

Proof. Set $y^0_{\tau,p,x_\tau,y_\tau}$ such that $0 = y^0_{\tau,p,x_\tau,y_\tau} + U_\tau(p|x_\tau, y_\tau)$ for some p, x and y. Then, by (1) and (3),

$$
\begin{aligned}
0 &= y^0_{\tau,p,x_\tau,y_\tau} + \sum_{\hat{n}_\tau=0}^{N_\tau-1} \sum_{\hat{n}_{-\tau}=0}^{N_{-\tau}} P^{N_\tau-1,N_{-\tau}}_{\hat{n},p} [\pi_\tau(\hat{n}_\tau + 1, \hat{n}_{-\tau}|x_\tau) + y_\tau - c_\tau] \\
&= \sum_{\hat{n}_\tau=0}^{N_\tau-1} \sum_{\hat{n}_{-\tau}=0}^{N_{-\tau}} P^{N_\tau-1,N_{-\tau}}_{\hat{n},p} [\pi_\tau(\hat{n}_\tau + 1, \hat{n}_{-\tau}|x_\tau) + y_\tau + y^0_{\tau,p,x_\tau,y_\tau} - c_\tau] \\
&= U_\tau(p|x_\tau, y_\tau + y^0_{\tau,p,x_\tau,y_\tau}),
\end{aligned}
$$

for any p and x. x and p remain unchanged throughout. □

A simple entry fee schedule, although it might be discriminatory, allows the seller to extract full rents without disturbing equilibrium p. Therefore, the optimal design problem in auctions with asymmetric potential bidders is equivalent to the maximization problem of the social surplus.

The social surplus associated with the transaction here is defined as the winning bidder's valuation for the item minus the sum of participation costs incurred by participants. Let $\beta(n|\cdot)$ be the incentive-compatible expected payment from the winning bidder to the

[8] Sufficiency for multiple equilibria is $\bar{p}_\tau \geq \tilde{p}_\tau$ for all τ. Intuitively, this is the case in which, if each party commits to assigning its maximum probability $\bar{p}_\tau$, there is no room for the other group to participate profitably.

seller in the auction taking n as given. Then the auction revenue $R(p|\cdot)$ is written as

$$R(p|x,y)=\sum_{\hat{n}_1=0}^{N_1}\sum_{\hat{n}_2=0}^{N_2}P^N_{\hat{n},p}[\beta(\hat{n}_1,\hat{n}_2|x)-y(\hat{n}_1,\hat{n}_2)].$$

From (1), the sum of the bidder's *ex ante* expected payoffs $U(p|x,y)=N_1p_1U_1(p|x_1,y_1)+N_2p_2U_2(p|x_2,y_2)$ is

$$U(p|x,y)=\sum_{\hat{n}_1=0}^{N_1}\sum_{\hat{n}_2=0}^{N_2}P^N_{\hat{n},p}[V(\hat{n}_1,\hat{n}_2|x)-\beta(\hat{n}_1,\hat{n}_2|x)+y(\hat{n}_1,\hat{n}_2)]-N_1p_1c_1-N_2p_2c_2,$$

where $V(\cdot)$ is the expected valuation of the winning bidder given $\hat{n}_1$, $\hat{n}_2$ and x. On the other hand, the total surplus $S(\cdot)$ is given by

$$\begin{aligned}S(p|x)&=R(p|x,y)+U(p|x,y)\\&=\sum_{\hat{n}_1=0}^{N_1}\sum_{\hat{n}_2=0}^{N_2}P^N_{\hat{n},p}V(\hat{n}_1,\hat{n}_2|x)-N_1p_1c_1-N_2p_2c_2.\end{aligned}\tag{6}$$

By lemma 1, $y^0_{\tau,p,x_\tau,y_\tau}$ enables the auctioneer to extract the bidder's rent at all for any p, *i.e.*, $U(p|x,y^0_{p,x,y})=0$. Hence, we obtain

$$S(p|x)=R(p|x,y^0_{p,x,y}).$$

This implies that, if the seller sets a monetary transfer scheme such that bidder's rent is zero, then the seller's revenue is identical to the social surplus for any mechanism.

In addition, (6) may remind us of the revenue equivalence theorem for asymmetric bidders with entry, as described in the following statement:

Theorem 1. *Suppose that the bidder's rent is fully extracted. If any two mechanisms have the same probability assignment functions and induce equal entry, then the two mechanisms generate the same revenue for the seller.*

This determines that the pure transfer y is redundant for the expected revenue as long as rents are fully extracted and the equilibrium entry p is unchanged. It follows that auctioneers have many alternative transfer schedules y that do not influence p.

In the reminder of this section, we explore the maximization problem over $S(p|x)$ to find the upper bound of social surplus $\hat{S}$. Among the arguments on S, we first investigate x and then control p to seek $\hat{S}$. It is trivially true that any *ex post* inefficiency stemming from distortive allocation or a positive reservation price decreases the social surplus.

Proposition 3. *Let x^* represent the ex post efficient mechanisms. For any $p \in [0,1] \times [0,1]$, the social surplus is maximized if and only if the mechanism is ex post efficient, namely for any p $S(p|x^*) > S(p|x)$ $\forall\ x \subset X \backslash x^*$.*

Proof. $V(n|x^*) \geq V(n|x)$ holds for any $x \subset X$. Hence, by (6), $S(p|x^*) \geq S(p|x)$ holds for any p. □

After focusing on *ex post* efficient mechanisms, we can discuss a useful theorem for equilibrium analysis as follows. This theorem provides a relationship between the *ex ante* payoff for each potential bidder and the expected marginal contribution to the social surplus of the bidder.

Theorem 2. *Let $v_\tau^{(1)}$ be the highest valuation among group τ bidders. Let $\phi_\tau(\cdot|x)$ be a matching function such that the bidder with $v_\tau^{(1)}$ and the bidder with $v_{-\tau}^{(1)}$ tie under some mechanism x if and only if $v_{-\tau}^{(1)} = \phi_\tau(v_\tau^{(1)}|x)$. Suppose that $\phi_\tau(v_\tau) \geq v_\tau$ for some τ. Then, for any n_τ,*

$$\begin{cases} \pi_1(n_1{+}1, n_2|x) = V(n_1{+}1, n_2|x) - V(n_1, n_2|x), \\ \pi_2(n_1, n_2{+}1|x) = V(n_1, n_2{+}1|x) - V(n_1, n_2|x), \end{cases} \tag{7}$$

if and only if $\phi_\tau(v_\tau) = v_\tau$.

See the Appendix for proof. This property is first introduced by Engelbrecht-Wiggans (1993) in an IPV setting. We extend the results to the asymmetric private value environment and give a sufficient condition for this property to hold. Most standard auctions satisfy the condition $\phi_\tau(v_\tau) \geq v_\tau$. For example, in asymmetric first-price auctions with a strong bidder and a weak bidder, we have $\phi_\tau(v_\tau) > v_\tau$, which suggests that (7) does not hold.[9] On the other hand, second-price auctions with a positive and constant reservation price satisfy $\phi_\tau(v_\tau) = v_\tau$. Such examples assure that *ex post* efficiency in allocation is not necessary for the marginal contribution theorem to hold as shown in Engelbrecht-Wiggans (1993). Obviously, the amount of contribution by one more additional bidder under the *ex post* inefficient mechanism is strictly smaller since $V(n_\tau{+}1, n_{-\tau}|x^*) - V(n_\tau, n_{-\tau}|x^*) > V(n_\tau{+}1, n_{-\tau}|x) - V(n_\tau, n_{-\tau}|x)$ for any $x \in X \backslash \{x^*\}$.

Now we return to the maximization problem of S. Taking a partial derivative of S

[9] Suppose that there are two bidders $j = \{1, 2\}$ whose value distribution is $F_j(v)$ on $[\bar{v}, \underline{v}]$ and $f_j(v)$ denotes the corresponding density. Assume that $lnF_j(v)$ is supermodular. Let β_j be the equilibrium bidding functions. The range of the equilibrium bidding functions should be identical so that $\beta_1(\bar{v}) = \beta_2(\bar{v})$ holds.

characterized in (6) with respect to p gives the first-order condition as follows

$$\begin{cases} \dfrac{\partial S}{\partial p_1} = N_1 \cdot \displaystyle\sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2} P_{\hat{n},p}^{N_1-1,N_2} [V(n_1+1, n_2|x) - V(n_1, n_2|x)] - N_1 c_1 \\ \dfrac{\partial S}{\partial p_2} = N_2 \cdot \displaystyle\sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2-1} P_{\hat{n},p}^{N_1,N_2-1} [V(n_1, n_2+1|x) - V(n_1, n_2|x)] - N_2 c_2. \end{cases} \tag{8}$$

Note that the second derivative is negative under the *ex post* efficient mechanism.[10] Then, by Theorem 2, (8) can be rewritten as

$$\begin{cases} \dfrac{\partial S}{\partial p_1} = N_1 \cdot \displaystyle\sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2} P_{\hat{n},p}^{N_1-1,N_2} [\pi_1(n_1+1, n_2|x) - c_1] \\ \dfrac{\partial S}{\partial p_2} = N_2 \cdot \displaystyle\sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2-1} P_{\hat{n},p}^{N_1,N_2-1} [\pi_2(n_1, n_2+1|x) - c_2]. \end{cases} \tag{9}$$

Recall that (9) coincides with $A_\tau(\cdot|x_\tau^*, 0)$. Therefore, (9) vanishes if $(p_1^*, p_2^*) \in (0,1) \times (0,1)$ and $\partial S(p|x^*, \cdot)/\partial p_\tau \leq 0$ or $\partial S(p|x^*, \cdot)/\partial p_\tau \geq 0$ if $p_\tau^* = 0$ or 1. This implies that the social gain from an incremental change in p_τ equals the social cost of an incremental change in the probability for each τ.[11] We thus establish the following proposition about efficient entry.

Proposition 4. *Suppose that a mechanism is ex post efficient and involves no transfer. Then, the bidder's profit-maximizing entry decision is the necessary and sufficient condition for efficient entry in the sense that the social gain from entry equals the cost of the entry.*

The result is quite reasonable, as we already know that the bidder's expected payoff is identical to the marginal contribution to the social surplus under x^*. Since the agent's problem is the same as the society's problem, the bidder's rational participation decision leads to an efficient outcome in the entry.

The sufficient condition for our maximization problem is a little more involved than that in the symmetric case. The following lemma demonstrates that, as a result of Hessian analysis over $S(p|x)$, any local maximum (or saddle) point on $S(p|x)$ is formed at the

[10] Show $\partial^2 S(p|x^*, \cdot)/\partial p_1^2 < 0$. Under the *ex post* efficient mechanism, $V(n_1+2, n_2|x^*, \cdot) - 2V(n_1+1, n_2|x^*, \cdot) + V(n_1, n_2|x^*, \cdot) = \pi_1(n_1+1, n_2|x^*, \cdot) - \pi_1(n_1, n_2|x^*, \cdot)$ holds. Furthermore, we obtain $\pi_1(n_1+1, n_2|x^*, \cdot) = \int_0^1 (1-\sigma) \frac{d}{d\sigma} v^s(\sigma) \sigma^{n_1} \cdot (\phi_1(\sigma))^{n_2} d\sigma$, where $\phi_\tau(\cdot) \equiv v_{-\tau}^{-1}(v_\tau(\sigma))$ is a matching function. Then, for any n_1 and n_2, $\pi_1(n_1+1, n_2|\cdot) - \pi_1(n_1, n_2|x^*, \cdot) = -\int_0^1 (1-\sigma)^2 \frac{d}{d\sigma} v_1(\sigma) \sigma^{n_1-1} \cdot (\phi_1(\sigma))^{n_2} d\sigma < 0$. The same is true for $\partial^2 S(p_1, p_2|x^*, \cdot)/\partial p_2^2 < 0$.

[11] In addition, if the gain is greater (or less) than the costs, deterministic participation (or staying out) occurs.

intersection between $A_\tau(\cdot)$ if the mechanism is *ex post* efficient without monetary transfer.

Lemma 2. *If $x = x^*$ and $y = \boldsymbol{0}$, then odd equilibria form a local maximum and even equilibria form a saddle point on the social welfare function $S(p|x^*)$.*

See the Appendix for proof. Again, the linkage between $A(\cdot)$ and $S(p|x^*)$ is shown here under the *ex post* efficient mechanisms with no transfer. The proof demonstrates that the gradient of $S(p|x^*)$ is determined by the positional relationship between $A_1(\cdot)$ and $A_2(\cdot)$.

If an analysis focuses on a symmetric equilibrium with *ex ante* symmetric bidders, efficiency and optimality in an entry are equivalent. Taking into account multiple equilibria, however, an efficient entry may not be the optimal entry. Different equilibria create different level of social surplus due to the difference in coordination costs associated with the randomness of the actual number of participants. Lemma 2 implies that *even* equilibria entail more waste of social welfare than *odd* equilibria and, hence, lower revenue. For instance, the symmetric equilibrium in a case of symmetric potential bidders suffers the highest coordination costs among three equilibria.

There is at least one *odd* equilibrium in asymmetric auctions with endogenous participation. In addition, in the case of multiple equilibria, one of the *odd* equilibria yields the highest social welfare. Hence, $\hat{S}$ is achieved at an odd equilibrium under *ex post* efficient mechanisms. The following proposition addresses the question of whether this occurs only in the case of *ex post* efficient mechanisms.

Proposition 5. *The social surplus created in an odd equilibrium attains $\hat{S}$ if the mechanism is ex post efficient and no transfer is used. Efficient entry is necessary for socially optimal entry, but not sufficient.*

Proof. Proof for the "*if*" part is obvious by proposition 3. The proof for the "*only if*" part is shown as follows. Let $\hat{p}^x$ denote the maximizer for $S(p|x)$ for any x. By construction, $S(\hat{p}^{x^*}|x^*) \geq S(p|x^*)$ for any p. By proposition 3, $S(\hat{p}^x|x^*) > S(\hat{p}^x|x)$ for any $x \neq x^*$. Hence, $S(\hat{p}^{x^*}|x^*) > S(\hat{p}^x|x)$ for any $x \neq x^*$. □

We conclude this section by considering implementation in the entry game with multiple equilibria. The auctioneer has two devices to influence entry *i.e.*, x and y. For example, setting a discriminatory reservation price which favors a group of bidders affects $A_\tau(\cdot)$ and, hence, the equilibrium formulation. In addition, bidding credits may help a particular group of potential bidders to participate more frequently and discourage the remaining bidders' participation. Furthermore, monetary transfer y enables the seller to control $A_\tau(\cdot)$ or extract more surplus from a group of bidders. The question of whether the auctioneer should choose any $x \neq x^*$ to reach the best equilibrium must then be asked.

The following lemma illustrates that the seller can implement any p as a unique equilibrium by controlling $y \in Y$. [12]

Lemma 3. *Given $x \in X$, there exists a set of transfer schedule $\hat{Y}_{p,x,y}$ with which any p will be induced as a unique mixed-strategy entry equilibrium.*

See the Appendix for proof. Recall lemma 1 insisting that full rent extraction is possible at any p by any $y^0_{p,x,y} \in Y^0_{p,x,y}$. Furthermore, we already know that $\hat{S}$ is never achieved unless the mechanism is *ex post* efficient in proposition 5. Hence, the optimal mechanism in auctions with asymmetric potential bidders involves a participation control, as reported in the following proposition.

Proposition 6. *The participation game has a unique mixed-strategy type-symmetric equilibrium in which the auctioneer's revenue is maximized if and only if the auctioneer employs the ex post efficient mechanism with an appropriately chosen transfer scheme.*

Proof. By lemma 3, there is a set of $\hat{y}$ that induces $\hat{p}$ as a unique odd equilibrium. Furthermore, if type τ potential bidders have positive expected rents, the auctioneer can extract them by setting $\hat{y}^0_\tau = \hat{y}_\tau + y^0_\tau$. Let $\hat{y}^0 = (\hat{y}^0_\tau, \hat{y}^0_{-\tau})$. Then, by lemma 3, $\hat{y}$ and $\hat{y}^0$ induce the same entry if and only if it is unique. □

5.1 Affiliated Private Value

Suppose that there exists $b_\tau(v|x)$ for some x such that the bidder who has the highest b wins the item. Define $\phi_\tau(v|x) = b^{-1}_{-\tau}(b_\tau(v|x)|x)$ and $F_\tau(v|x) = F_\tau(\phi_{-\tau}(v|x))$. Then, (2) becomes

$$\begin{aligned}\pi_1(n_1, n_2|x) &= \int_z \int_v (1 - F_1(v|z))(F_1(v|z))^{n_1 - 1}(F_2(v|z))^{n_2} dv g(z) dz,\\ &= \int_z \int_v v[(F_1(v|z))^{n_1}(F_2(v|z))^{n_2}]' dv dz - \int_z \int_v v[(F_1(v|z))^{n_1 - 1}(F_2(v|z))^{n_2}]' dv g(z) dz\\ &= V(n_1, n_2|x^*) - V(n_1 - 1, n_2|x^*),\end{aligned}$$

which extends theorem 2 to an affiliated private-value (APV) paradigm. Now, we have extended the results to the asymmetric APV environment.

6 Discussion

We show that, if endogenous participation is accounted for, *ex post* efficiency is necessary for an optimal mechanism. The results contradict the existing theorems for optimal design

[12] This is primarily because y is a function of n_1 and n_2 rather than a negative constant variable *i.e.*, a fixed entry fee. However, nothing would be gained in the symmetric model if y is a function of n.

with a fixed set of asymmetric bidders, which insist that *ex post* efficient mechanisms are not optimal (See Myerson (1981), McAfee and McMillan (1989), and Bulow and Roberts (1989)). Our model illustrates that the costs of introducing a distortive assignment rule for rent extraction always outweigh the benefits due to a serious inefficiency in participation.

Proposition 6 also impacts the ranking theorems with IPV asymmetric auctions. Vickrey (1961) showed that there is no general ranking between first and second-price asymmetric auctions, if the number of bidders is exogenously determined.[13] Considering endogenous participation, we obtain a clear ranking. Second-price mechanisms always dominate first-price mechanisms. In addition, Milgrom and Weber (1982) suggests that second-price mechanisms yield higher revenue than first-price ones if signals are affiliated. Therefore, our proposition about the superiority of second-price mechanisms still holds in the case of asymmetric APV environments.

The proposition in which efficient entry is a necessary condition for optimal outcome is also referred to in Levin and Smith (1994).[14] They address the issue that when more potential bidders exceed the point of transition between pure- and mixed-entry strategies, the result is a waste of social welfare.[15] Our approach is a non-trivial extension from Levin and Smith (1994), since we provide a generalized scheme that makes it possible to evaluate revenue across equilibria with asymmetric potential bidders.

For ways of promoting competition in asymmetric auctions with participation, our analysis relates to Ayres and Cramton (1996) and Gilbert and Klemperer (2000). Motivated by auctioneers' concern about insufficient competition among well-qualified bidders, their studies explore whether subsidizing weak buyers through a distortive allocation rule increases revenue. Both conclude that promoting entry by weak buyers will enhance revenue. Since an *even* equilibrium is a saddle point on $S(\cdot|x^*)$, the change of an allocation

[13] The existence of an equilibrium bidding function in asymmetric first-price auctions with a fixed number of bidders is shown by Lebrun (1999). In general, weak bidders would bid more aggressively than strong bidders in the first-price auction, resulting in *ex post* inefficient allocation. See Maskin and Riley (2000).

[14] Ye (2004) also shows the advantage of asymmetric equilibrium by using Jensen's inequality. However, the results are crucially dependent upon the assumption that potential bidders are *ex ante* identical.

[15] This argument is true without a symmetry assumption if the mechanisms are *ex post* efficient. Since $S(p_1,p_2|x,N_1,N_2) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2} P_{\hat{n},p}^{N_1-1,N_2}[p_1 \cdot V(n_1+1,n_2|x) + (1-p_1)\cdot V(n_1,n_2|x)] - N_1p_1c_1 - N_2p_2c_2 = \sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2-1} P_{\hat{n},p}^{N_1,N_2-1}[p_2 \cdot V(n_1,n_2+1|x) + (1-p_2)\cdot V(n_1,n_2|x)] - N_1p_1c_1 - N_2p_2c_2$, $S(p_1,p_2|x,N_1-1,N_2) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2} P_{\hat{n},p}^{N_1,N_2-1} V(n_1,n_2|x) - (N_1-1)p_1c_1 - N_2p_2c_2$ and$S(p_1,p_2|x,N_1,N_21) = \sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2-1} P_{\hat{n},p}^{N_1,N_2-1} V(n_1,n_2|x) - N_1p_1c_1 - (N_2-1)p_2c_2$ for any p_1, p_2, and x, we have

$$\begin{cases} S(p|x^*,N_1,N_2) - S(p|x^*,N_1-1,N_2) = p_1 \cdot \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2} P_{\hat{n},p}^{N_1-1,N_2}[V(n_1+1,n_2|\cdot) - V(n_1,n_2|\cdot)] - p_1 \cdot c_1 \\ S(p|x^*,N_1,N_2) - S(p|x^*,N_1,N_2-1) = p_2 \cdot \sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2-1} P_{\hat{n},p}^{N_1,N_2-1}[V(n_1,n_2+1|\cdot) - V(n_1,n_2|\cdot)] - p_2 \cdot c_2. \end{cases}$$

If $x = x^*$, then, by theorem 2, both are equal to zero. This indicates that, if a potential bidder, regardless of its type, is eliminated and the remaining potential bidders still use p, then social welfare is unchanged. If the remaining potential bidders choose p' according to their new best response, which accounts one fewer potential bidders, then, social surplus increases by construction.

rule from x^* to some x' may induce another p, which creates greater S despite some efficiency loss in allocation. However, our study suggests that the outcome is sub-optimal. The first best outcome is achieved only through an *ex post* efficient allocation with a pure transfer.

A transfer may often be seen in the real-world procurement auction as the requirement for a higher financial guarantee for bidders. It is costly for bidders but beneficial to the auctioneer by reducing the risk of facing default. Government spending for improving small business access can also be considered to be a transfer. An important aspect of participation control, which includes an implicit one, is that any change in participation from the optimal equilibrium results in both efficiency loss due to coordination costs and fluctuation in participation.

7 Conclusion

Over the past decade, the model of auctions with endogenous participation has provided a striking result, namely, that both efficiency and revenue maximization can be achieved simultaneously. Despite the contribution of endogenous participation models, they depend heavily on the assumption that the potential bidders are *ex ante* the same. Little progress has been made in the theory of auction with asymmetric endogenous entry.

The relaxation of symmetric assumption is not a trivial extension from the existing symmetric model. First, with symmetric bidders, optimal design problems boil down to the optimal choice of a reservation price, as investigated in Riley and Samuelson (1981). Introducing asymmetry, the optimal design problem becomes more complicated. As discovered in Myerson (1981), the appropriately chosen distortive mechanisms enhance revenue. This proposition, however, absolutely ignores the effects of a potential bidder's participation. Accounting for the rational decision of a potential bidder's participation, any rent extraction by distortive allocation causes inefficient entry, and, hence, simple auctions are optimal. The results provide a new interpretation for the widespread use of simple auctions.

Second, the endogenous entry model becomes applicable to a more general environment. In procurement auctions for international projects, for example, a number of contractors randomly participate. Based on the fact that traveling costs as well as currency values differ from country to country, it is impossible to suppose that all potential bidders are *ex ante* the same. Almost all empirical models for auctions with endogenous participation, so far, have been based on the symmetric assumption (*e.g.* Li and Zheng (2006)). We hope that our model contributes to the enrichment of the empirical analysis for auctions with endogenous participation.

Appendix

Proof for Proposition 1

Define $H_\tau(p_{-\tau}|\cdot) : [0,1] \to [0,1]$ as the solution of (4) for p_τ. Since U^i_τ is decreasing in p, we have $p'_{-\tau} \le p''_{-\tau} \le p'''_{-\tau}$, where $p'_\tau \in \{p_{-\tau}|U^i_\tau(1, p_{-\tau}|\cdot) > 0\}$, $p''_{-\tau} \in [0,1]\backslash\{p_{-\tau}|U^i_\tau(1, p_{-\tau}|\cdot) > 0\} \cup \{p_{-\tau}|U^i_\tau(0, p_{-\tau}|\cdot) < 0\}$, and $p'''_{-\tau} \in \{p_{-\tau}|U^i_\tau(0, p_{-\tau}|\cdot) < 0\}$. Therefore, H_τ can be described as

$$H_\tau(p_{-\tau}|\cdot) = \begin{cases} 1 & \text{if } \{p_{-\tau}|U^i_\tau(1, p_{-\tau}|\cdot) > 0\} \\ 0 & \text{if } \{p_{-\tau}|U^i_\tau(0, p_{-\tau}|\cdot) < 0\} \\ \{p_\tau|U^i_\tau(p_1, p_2|\cdot) = 0\} & \text{otherwise.} \end{cases} \tag{10}$$

Since (p_1^*, p_2^*) must satisfy (10), we have

$$\begin{cases} H_1(p_2^*|x_1, y_1) = p_1^*, \\ H_2(p_1^*|x_2, y_2) = p_2^*. \end{cases}$$

In other words, if we define $H(p|\cdot) = \times_{\tau\in\{1,2\}} H_\tau(p_{-\tau}|\cdot)$, a type-symmetric equilibrium $p^* = (p_1^*, p_2^*)$ is a fixed point of H *i.e.*, $p^* \in H(p^*|x, y)$.

$u_\tau(n|\cdot)$ is decreasing in n. Since n follows a binomial distribution, U_τ is continuous. Hence, by (10), $H_\tau(p_{-\tau}|\cdot)$ is continuous. For each τ the set $H_\tau(p_{-\tau})$ is nonempty and has a closed graph since $H_\tau(\cdot)$ is continuous. Thus, by the fixed-point theorem, there exists at least one fixed point on H. A mixed-strategy equilibrium is a fixed point.

Proof for Proposition 2

By the characteristic of binomial distribution, the following identities hold:

$$\begin{cases} \sum_{\hat{n}_1=0}^{N_1-1}\sum_{\hat{n}_2=0}^{N_2} P_{\hat{n},p}^{N_1-1,N_2} u_1(\hat{n}_1+1,\hat{n}_2|\cdot) \\ \qquad \equiv \sum_{\hat{n}_1=0}^{N_1-1}\sum_{\hat{n}_2=0}^{N_2-1} P_{\hat{n},p}^{N_1-1,N_2-1}[p_2\cdot u_1(\hat{n}_1+1,\hat{n}_2+1|\cdot)+(1-p_2)\cdot u_1(\hat{n}_1+1,\hat{n}_2|\cdot)] \\ \sum_{\hat{n}_1=0}^{N_1}\sum_{\hat{n}_2=0}^{N_2-1} P_{\hat{n},p}^{N_1,N_2-1} u_2(\hat{n}_1,\hat{n}_2+1|\cdot) \\ \qquad \equiv \sum_{\hat{n}_1=0}^{N_1-1}\sum_{\hat{n}_2=0}^{N_2-1} P_{\hat{n},p}^{N_1-1,N_2-1}[p_1\cdot u_2(\hat{n}_1+1,\hat{n}_2+1|\cdot)+(1-p_1)\cdot u_2(\hat{n}_1,\hat{n}_2+1|\cdot)]. \end{cases} \tag{11}$$

If $U_1 = U_2$, then by monotonicity,

$$\begin{aligned} p_2 \sum_{\hat{n}_1}\sum_{\hat{n}_2} P_{\hat{n}} u_1(\hat{n}_1+1,\hat{n}_2+1) + (1-p_2)\sum_{\hat{n}_1}\sum_{\hat{n}_2} P_{\hat{n}} u_1(\hat{n}_1+1,\hat{n}_2) \\ \leq p_1 \sum_{\hat{n}_1}\sum_{\hat{n}_2} P_{\hat{n}} u_1(\hat{n}_1+1,\hat{n}_2+1) + (1-p_1)\sum_{\hat{n}_1}\sum_{\hat{n}_2} P_{\hat{n}} u_1(\hat{n}_1+1,\hat{n}_2). \end{aligned}$$

Hence, $(p_2-p_1)\sum_{\hat{n}_1}\sum_{\hat{n}_2} P_{\hat{n}} u_1(\hat{n}_1+1,\hat{n}_2+1) \leq (p_2-p_1)\sum_{\hat{n}_1}\sum_{\hat{n}_2} P_{\hat{n}} u_1(\hat{n}_1+1,\hat{n}_2)$. Since, for any p_1 and p_2, $\sum_{\hat{n}_1}\sum_{\hat{n}_2} P_{\hat{n}} u_1(\hat{n}_1+1,\hat{n}_2+1) - \sum_{\hat{n}_1}\sum_{\hat{n}_2} P_{\hat{n}} u_1(\hat{n}_1+1,\hat{n}_2) < 0$, one must obtain $p_2 \geq p_1$. Equality holds if both $u_1(n_1+1,n_2+1) = u_2(n_1+1,n_2+1)$ and $u_1(n_1+1,n_2) = u_2(n_1,n_2+1)$ hold for all n.

Proof for Lemma 3

Suppose, by contradiction, that the set $\hat{Y}$ is empty. Set $y_1 = \tilde{y}_1$ such that $u_1(n_1+1,n_2+1|x_1,\tilde{y}_1) - u_1(n_1+1,n_2|x_1,\tilde{y}_1) = 0$ and $u_1(n_1+2,n_2|x_1,\tilde{y}_1) - u_1(n_1+1,n_2|x_1,\tilde{y}_1) > 0$. Then, $G_1(p|x_1,\tilde{y}_1) = 0$ for any p. Moreover, set $y_2 = \tilde{y}_2$ such that $u_2(n_1,n_2+2|x_2,\tilde{y}_2) - u_2(n_1,n_2+1|x_2,\tilde{y}_2) > 0$ and $u_2(n_1+1,n_2+1|x_2,\tilde{y}_2) - u_2(n_1+1,n_2|x_2,\tilde{y}_2) = 0$. Then, $G_2(p|x_2,\tilde{y}_2) \to \infty$ for any p. Since $G_1 < G_2$, $\tilde{y} = (\tilde{y}_1,\tilde{y}_2) \in \hat{Y}$, we have reached a contradiction. Thus $\hat{Y}$ is nonempty.

Let $\hat{y} \in \hat{Y}$. Set $y_\tau^0(p) = -U_\tau(p|x_\tau,\hat{y}_\tau)$ for some p. Then, $U_\tau(p|x_\tau,\hat{y}_\tau + y_\tau^0(p)) = 0$ for any p. Since $y_\tau^0(p)$ is constant for all n, $G_1(p|x_1,\hat{y}_1+y_1^0) < G_2(p|x_2,\hat{y}_2+y_2^0)$ for any x. Hence, $\hat{y}(p) + y^0(p) \in \hat{Y}$.

Proof for Theorem 2

Proof. Suppose that there exists $b_\tau(v|x)$ for some x such that the bidder who has the highest b wins the item. Define $\phi_\tau(v|x) = b^{-1}_{-\tau}(b_\tau(v|x)|x)$ and $\hat{F}_\tau(v|x) = F_\tau(\phi_{-\tau}(v|x))$. Then, (2) becomes

$$\begin{aligned}\pi_1(n_1,n_2|x) &= \int_v (1-F_1(v))(F_1(v))^{n_1-1}(\hat{F}_2(v|x))^{n_2}dv,\\ &= \int_v (F_1(v))^{n_1-1}(\hat{F}_2(v|x))^{n_2}dv - \int_v (F_1(v))^{n_1}(\hat{F}_2(v|x))^{n_2}dv,\\ &= \int_v v[(F_1(v))^{n_1}(\hat{F}_2(v|x))^{n_2}]'dv - \int_v v[(F_1(v))^{n_1-1}(\hat{F}_2(v|x))^{n_2}]'dv.\end{aligned}$$

where $F_\tau(\cdot) = v^{-1}_\tau(\cdot)$ and $f_\tau(\cdot) = F'_\tau(\cdot)$. Integral by parts yields the last equality. Since $\hat{F}_1(v|x) = F_1(\phi_2(v))$, $\hat{F}_1(\phi_1(v)) = F_1(\phi_2(\phi_1(v))) = F_1(v)$ and $\phi_2(\phi_1(v)) = v_1(v)$, letting $\phi_1(v) = \hat{v}$, it is possible to obtain

$$\begin{aligned}\int v(F_1(v))^{n_1}[(\hat{F}_2(v|x))^{n_2}]'dv &= \int \phi_2(\phi_1(v))\cdot[\hat{F}_1(\phi_1(v))]^{n_1}n_2f_2(\phi_1(v))(F_2(\phi_1(v)))^{n_2-1}\phi'_1(v)dv\\ &= \int_{\hat{v}} \phi_2(\hat{v})\cdot[\hat{F}_1(\hat{v})]^{n_1}n_2f_2(\phi_1(v))(F_2(\phi_1(v)))^{n_2-1}d\hat{v}.\end{aligned}$$

Recall

$$\begin{aligned}V(n_1-1,n_2|x) &= \int v(n_1-1)f_1(v)(F_1(v))^{n_1-2}(\hat{F}_2(v|x))^{n_2}dv + \int v(\hat{F}_1(v|x))^{n_1-1}n_2f_2(v)(F_2(v))^{n_2-1}dv\\ V(n_1,n_2|x) &= \int vn_1f_1(v)(F_1(v))^{n_1-1}(\hat{F}_2(v|x))^{n_2}dv + \int v(\hat{F}_1(v|x))^{n_1}n_2f_2(v)(F_2(v))^{n_2-1}dv.\end{aligned}$$

Therefore,

$$V(n_1,n_2|x) - V(n_1-1,n_2|x) + \int_v [\phi_2(v)-v][1-\hat{F}_1(v|x)](\hat{F}_1(v|x))^{n_1-1}[(F_2(v))^{n_2}]'\,dv = \pi_1(n_1,n_2|x).$$

$\phi_\tau(v) = v$ for any v under the *ex post* efficient mechanism. Therefore, the third term on the left-hand side vanishes if $x = x^*$. Clearly, the term is typically non-zero for any $x \subset X\backslash\{x^*\}$. □

Proof for Lemma 2

Since $\partial^2 S(p|x^*,\cdot)/\partial p^2 < 0$, the first-order principle minor of the Hessian on $S(p|\cdot)$ is negative.

Take the second-order derivative on S.

$$\frac{\partial^2 S(p|\cdot)}{\partial p_1^2} = N_1(N_1-1)\cdot\sum_{\hat{n}_1=0}^{N_1-2}\sum_{\hat{n}_2=0}^{N_2} P_{\hat{n},p}^{N_1-2,N_2}[V(n_1+2,n_2|\cdot) - 2V(n_1+1,n_2|\cdot) + V(n_1,n_2|\cdot)]$$

$$\frac{\partial^2 S(p|\cdot)}{\partial p_1 \partial p_2} = N_1 N_2\cdot\sum_{\hat{n}_1=0}^{N_1-1}\sum_{\hat{n}_2=0}^{N_2-1} P_{\hat{n},p}^{N_1-1,N_2-1}[V(n_1+1,n_2+1|\cdot)-V(n_1,n_2+1|\cdot) - V(n_1+1,n_2|\cdot)+V(n_1,n_2|\cdot)]$$

$$\frac{\partial^2 S(p|\cdot)}{\partial p_2^2} = N_2(N_2-1)\cdot\sum_{\hat{n}_1=0}^{N_1}\sum_{\hat{n}_2=0}^{N_2-2} P_{\hat{n},p}^{N_1,N_2-2}[V(n_1,n_2+2|\cdot) - 2V(n_1,n_2+1|\cdot) + V(n_1,n_2|\cdot)]$$

The second-order principal minor of the Hessian on $S(p_1,p_2|\cdot)$, which is given as $(N_1-1)(N_2-1)\cdot\sum_{\hat{n}_1=0}^{N_1-2}\sum_{\hat{n}_2=0}^{N_2}P_{\hat{n},p}^{N_1-2,N_2}[\pi_1(n_1,n_2+2|x)-\pi_1(n_1,n_2+1|x)]\cdot\sum_{\hat{n}_1=0}^{N_1}\sum_{\hat{n}_2=0}^{N_2-2}P_{\hat{n},p}^{N_1,N_2-2}[\pi_2(n_1+2,n_2|x)-\pi_2(n_1+1,n_2|x)]-N_1N_2\cdot\sum_{\hat{n}_1=0}^{N_1-1}\sum_{\hat{n}_2=0}^{N_2-1}P_{\hat{n},p}^{N_1-1,N_2-1}[\pi_1(n_1,n_2+1|x)-\pi_1(n_1,n_2|x)]\cdot\sum_{\hat{n}_1=0}^{N_1-1}\sum_{\hat{n}_2=0}^{N_2-1}P_{\hat{n},p}^{N_1-1,N_2-1}[\pi_2(n_1+1,n_2|x)-\pi_2(n_1,n_2|x)]$ by Lemma 2, is positive if and only if $G_1(p|\cdot)-G_2(p|\cdot) > 0$. Hence, the even equilibrium is a saddle point. Finally, we show that $S(p_1,p_2|x^*, U_1(p_1,p_2)=0)$ is increasing in p_1 if and only if $U_1(p_1,p_2|x^*) > U_2(p_1,p_2|x^*)$. Suppose, by contradiction, that there exist p_1' and p_2' such that $S(p_1',p_2'|x^*)|_{A_1=0}$ is decreasing in p_1. Since $U_2(\cdot)$ is continuous in p_2, there exists some $p_2'' < p_2'$ such that $U_2(p_1',p_2'')=0$. By (8), $S(p_1',p_2''|x^*) > S(p_1',p_2|x^*)$ for any p_2. Therefore, $S(p_1',p_2''|x^*) > S(p_1',p_2'|x^*)$. Since $U_1(\cdot)$ is continuous in p_1, there exists $p'' > p'$ such that $U_1(p_1'',p_2'')=0$. By (8), $S(p_1'',p_2''|x^*) > S(p_1,p_2''|x^*)$ for any p_1. Therefore, $S(p_1'',p_2''|x^*) > S(p_1',p_2''|x^*)$. Hence, $S(p_1'',p_2''|x^*) > S(p_1',p_2'|x^*)$, which leads to contradiction.

References

Athey, Susan and Philip A. Haile, "Empirical Models of Auctions," Levine's Bibliography 122247000000001045, UCLA Department of Economics March 2006.

Ayres, Ian and Peter Cramton, "Deficit Reduction Through Diversity: How Affirmative Action at the FCC Increased Auction Competition," Technical Report 1996.

Bulow, Jeremy and John Roberts, "The Simple Economics of Optimal Auctions," *Journal of Political Economy*, October 1989, *97* (5), 1060–90.

Engelbrecht-Wiggans, Richard, "Optimal Auctions Revisited," *Games and Economic Behavior*, April 1993, *5* (2), 227–239.

Gilbert, Richard J. and Paul Klemperer, "An Equilibrium Theory of Rationing," *RAND Journal of Economics*, Spring 2000, *31* (1), 1–21.

Kjerstad, Egil and Steinar Vagstad, "Procurement auctions with entry of bidders," *International Journal of Industrial Organization*, December 2000, *18* (8), 1243–1257.

Lebrun, Bernard, "First Price Auctions in the Asymmetric N Bidder Case," *International Economic Review*, February 1999, *40* (1), 125–42.

Levin, Dan and James L Smith, "Equilibrium in Auctions with Entry," *American Economic Review*, June 1994, *84* (3), 585–99.

Li, Tong and Xiaoyong Zheng, "Entry and competition effects in first-price auctions: theory and evidence from procurement auctions," CeMMAP working papers CWP13/06, Centre for Microdata Methods and Practice, Institute for Fiscal Studies August 2006.

Maskin, Eric and John Riley, "Asymmetric Auctions," *Review of Economic Studies*, July 2000, *67* (3), 413–38.

McAfee, R. Preston and John McMillan, "Auctions with entry," *Economics Letters*, 1987, *23* (4), 343–347.

_ and _ , "Government procurement and international trade," *Journal of International Economics*, May 1989, *26* (3-4), 291–308.

Milgrom, Paul R and Robert J Weber, "A Theory of Auctions and Competitive Bidding," *Econometrica*, September 1982, *50* (5), 1089–1122.

Myerson, Roger B, "Optimal auction design," *Mathematics of Operations Research*, 1981, *6* (1), 58–73.

Pevnitskaya, Svetlana, "Endogenous Entry and Self-selection in Private Value Auctions: An Experimental Study," 2003. working paper,California Institute of Technology, Division of the Humanities and Social Sciences.

_ , "Endogenous Entry in First-Price Private Value Auctions: the Self-Selection Effect," 2003. working paper,Ohio State University.

Riley, John G and William F Samuelson, "Optimal Auctions," *American Economic Review*, June 1981, *71* (3), 381–92.

Vickrey, William, "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, Mar 1961, *16* (1), 8–37.

Ye, Lixin, "Optimal Auctions with Endogenous Entry," *Contributions to Theoretical Economics*, 2004, *4* (1), 1158–1158.